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# Kalman reduced form and pole placement by state feedback for multi-input linear systems over Hermite rings

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A Kalman reduced form is obtained for linear systems over Hermite rings. This reduced form gives information of the set of assignable polynomials to a given linear system.

**KEYWORDS**

assignable polynomial, feedback, Hermite ring, linear system

**MSC CLASSIFICATION**

93B55; 13P25

## 1 | INTRODUCTION

Linear control systems over commutative rings are a generalization of linear control systems used in the study of parametric families of linear systems,[1–3] or in the study of convolutional codes over finite rings.[4,5]

Reachability is a central notion in the feedback control of linear systems. It is well known that every reachable linear system over a field is pole-assignable. But this result does not generalize to arbitrary commutative rings $R$. The study of commutative rings where every reachable system is pole-assignable, that is, PA-rings, goes back, among several classic references, to seminal memoirs by Brewer, Bunce, and van Vleck[1] and by Sontag.[2] The problem of characterizing the class of PA-rings is still open.

On the other hand, it is interesting to know what polynomials can be assigned in the general case, that is, to non-reachable linear systems. It is well known that Kalman reduced form gives a solution in the case of linear systems over fields: There exists an invariant monic polynomial associated to the system, and we can assign exactly those monic multiples of the invariant polynomial.

A similar result is given[6] for single input linear systems over a Bézout domain. It is even available an algorithm to test if some polynomial is assignable to a given linear system.

This paper is devoted to prove similar pole-shifting results on Hermite rings both for single-input and multi-input linear systems. A Kalman-like reduced form is obtained for every linear system over an Hermite ring $R$. In the case of $R$ being also a Bézout domain then further information about the set of assignable polynomials is obtained.

The class of Hermite rings contains principal ideal domains like rational integers $\mathbb{Z}$ and polynomials on a single indeterminate $\mathbb{K}[x]$. It also contains modular integer rings $\mathbb{Z}/n\mathbb{Z}$ which are plenty of zero divisors, and hence, they are not domains. Finally, valuation rings like finite hyperreals[7] and power series algebras $\mathbb{K}[[x]]$ are also Hermite rings. Some regulation problems of cyber-security systems[8] might be faced as regulation problems on linear systems with infinitesimal inputs, that is, over the ring of finite hyperreals,[9,10] or over some Krylov hyperreal space.[11] Classical reference for

**Abbreviation:** PA, pole assignability.

Hermite rings is Kaplansky's seminal work.[12] Some recent references[13] deal with remarkable classes of Hermite rings. Bézout domains are also of interest recently.[14]

The paper is organized as follows. Section 2 deals with well-known facts and definitions about pole placement and pole assignment of linear systems over commutative rings. Equivalence of linear systems is also reviewed. Section 3 contains the main result of the paper: a Kalman-like normal form over Hermite rings. Next, Section 4 is devoted to apply the result to the problem of finding the class of polynomials that can be assigned to a given (non-reachable) linear system. Finally, our conclusions are given in Section 5. Special attention is payed to the regulator problem for linear systems with outputs.[8,15]

## 2 | LINEAR SYSTEMS, POLE PLACEMENT, AND FEEDBACK ACTIONS

Throughout this paper, all rings $R$ will be commutative and with unit element 1. The characteristic polynomial $\det(z\mathbf{I}-M)$ of a square matrix $M \in R^{n\times n}$ is often denoted by $\chi(M)$. The $j$th determinantal ideal of a rectangular matrix $L$, that is, the ideal of $R$ generated by all $j \times j$ minors of matrix $L$, is denoted by $\mathcal{U}_j(L)$.

An $m$-input, $n$-dimensional linear system over $R$, or shortly a system of size $(n, m)$, is a pair $\Sigma = (A, B)$ of matrices over $R$, where $A \in R^{n\times n}$ and $B \in R^{n\times m}$.

Block matrix $A^*B = [B|AB| \dots |A^{n-1}B]$ is called the reachability matrix of $(A, B)$. The linear system $(A, B)$ is called reachable if the columns of $A^*B$ span $R^n$, that is, $\mathrm{Im}(A^*B) = R^n$.

### 2.1 | A review of some pole placement results

The *Classical pole-shifting theorem over fields* $\mathbb{K}$ states that if $\Sigma = (A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$ is a reachable system of size $(n, m)$ then for every monic polynomial $p(z) = z^n + a_{n-1}z^{n-1} + \dots + a_1 z + a_0$ of degree $n$, there exists a matrix $F \in \mathbb{K}^{m\times n}$ (a feedback matrix) such that $\chi(A + BF) = p(z)$. Hence, we can assign any polynomial by a suitable feedback matrix.

Denote by $\mathrm{Pols}(A, B)$ the set of assignable polynomials to a general system $(A, B) \in \mathbb{K}^{n\times n} \times \mathbb{K}^{n\times m}$, that is to say a monic polynomial $p(z)$ of degree $n$ is in $\mathrm{Pols}(A, B)$ if and only if there exists a matrix $F \in \mathbb{K}^{m\times n}$ such that $\chi(A + BF) = p(z)$. It is known that if the system $\Sigma = (A, B)$ is not reachable then some polynomials are not assigned by feedback.

*Remark* 1. Kalman's decomposition of linear systems over fields $\mathbb{K}$ yields that there exists a monic polynomial $f(z)$, whose degree equals $n - \mathrm{rank}(A^*B)$, such that $p(z)$ is in $\mathrm{Pols}(A, B)$ if and only if[16] $p(z)$ is a multiple of $f(z)$.

*Remark* 2. Kalman's decomposition is in fact valid over any commutative ring $R$ for linear systems whose reachability subspace is a direct summand of the state-space.[16] Therefore above result is valid over any PA ring for linear systems whose reachability subspace are direct summands of the state-space. The existence of a Kalman-like decomposition of linear time-invariant systems over commutative rings has been studied from the coefficient assignability point of view by Sáez-Schwedt and Sánchez-Giralda.[17]

### 2.2 | Single input systems

Single input linear systems are those linear systems of size $(n, 1)$, that is to say, pairs $(A, b) \in R^{n\times n} \times R^{n\times 1}$. A single input system over $R$ is pole assignable if and only if it is reachable,[1] that is to say, pole shifting theorem is valid for single input systems over any commutative ring.

In the case of non-reachable single input systems over a Bézout domain $R$ there is a procedure[6] to check if some concrete polynomial is feedback assignable to the system. A Kalman-like result is also available. Every single input system over a Bézout domain splits in a weakly reachable system together with a zero control system.[6] For general Hermite rings, above membership problem is still open.

### 2.3 | Equivalence of linear systems

Two systems $(A, B)$ and $(A', B')$ of the same size are called algebraically equivalent if there exist invertible matrices $P, Q$ of appropriate sizes such that $A' = PAP^{-1}$ and $B' = PBQ$. If feedback actions are also allowed, $(A, B)$ and $(A', B')$ are feedback equivalent if $A' = PAP^{-1} + PBF$ and $B' = PBQ$, with $P, Q$ invertible matrices, and $F$ a suitable (feedback) matrix.

*Remark* 3. Equivalent systems have equal sets of assignable polynomials.[18] That is to say, if system $(A, B)$ is equivalent (algebraic or feedback) to system $(A', B')$ then $\mathrm{Pols}(A, B) = \mathrm{Pols}(A', B')$. Therefore we will use reduced forms of linear systems in order to obtain sets of assignable polynomials.

## 3 | A REDUCED FORM FOR LINEAR SYSTEMS OVER HERMITE RINGS

Let $R$ be an integral domain ($ab = 0$ implies $a = 0$ or $b = 0$) and let $\mathbb{K}$ be its field of fractions. It makes sense to define the rank of a matrix $M$ over $R$ as the rank of $M$ over $\mathbb{K}$. It is also possible to consider any system $(A, B)$ over $R$ as a linear system over $\mathbb{K}$ by means of scalar extension. System $(A, B)$ is called weakly reachable[19] if $(A, B)$ is reachable over $\mathbb{K}$, or equivalently, if the ideal of maximal minors of $A^*B$ is not zero.

A Bézout domain $R$ is an integral domain such that all finitely generated ideals are principal: given two elements $a, b$ in $R$, there exists a greatest common divisor $d$ and elements $x$, $y$ in $R$ satisfying the Bézout identity $ax + by = d$.

A commutative ring $R$ is called Hermite, in the sense of Kaplansky,[12] if every matrix is lower triangulable. The notions of Bézout and Hermite are equivalent for domains. In the case of rings with zero-divisors, there are Bézout rings that are not Hermite, but they are rather hard to construct.[20]

Let $R$ be a Hermite ring, then every rectangular matrix $B \in R^{n\times m}$ can be put (after left multiplication by an invertible matrix $P$) into row echelon form. Recall that a matrix is in row echelon form if the following condition hold: Denote by $d_i$ the first nonzero element of row $i$ in $PB$, then there are only zeros below and to the left of the "pivot" $d_i$. A typical example could be

$$\left[\begin{array}{ccccccc} 0 & d_1 & * & * & * & * & * \\ 0 & 0 & d_2 & * & * & * & * \\ 0 & 0 & 0 & 0 & 0 & d_3 & * \\ \hline 0 & 0 & 0 & 0 & 0 & 0 & 0 \end{array}\right].$$

We say that a row echelon form is "full row echelon form" if all rows are nonzero.

An $n \times m$ matrix $D$ over a commutative ring $R$ is said to satisfy the property (*) if one can extract $n$ columns of $D$ forming a upper triangular block

$$\begin{bmatrix} d_1 & & * \\ & \ddots & \\ 0 & & d_n \end{bmatrix},$$

with $d_1, \dots, d_n$ nonzero (note that in particular $n \le m$). Next, we show that the above property is preserved by left multiplication by a full row rank matrix in echelon form, if the ring $R$ is an integral domain.

**Theorem 1.** *Let $R$ be an integral domain and consider a matrix $D \in R^{n\times m}$ in full row rank echelon form and a matrix $E \in R^{m\times p}$ with the property (*). Then, $DE$ also satisfies the property (*).*

*Proof.* For each $i = 1, \dots, n$, the row $i$ of $D$ has a first nonzero element $d_i$ in some position $k_i$, $i \le k_i \le m$. By the property (*), $E$ has some column, say $c_i$, of the form $c_i = [*, \dots, *, e_{k_i}, 0, \dots, 0]^t$, with $e_{k_i} \neq 0$ in position $k_i$. Moreover, for each $j > i$, the row $j$ of $D$ has zeros in the first $k_i$ positions, therefore one has $Dc_i = [*, \dots, *, d_i e_{k_i}, 0, \dots, 0]^t$, with $d_i e_{k_i} \neq 0$ in position $i$. It follows that matrix $DE$ contains $n$ columns of the form:

$$D[c_1| \dots |c_n] = \begin{bmatrix} d_1 e_{k_1} & & * \\ & \ddots & \\ 0 & & d_n e_{k_n} \end{bmatrix}.$$

Since $R$ has no zero divisors, all the elements in the main diagonal are nonzero, which proves that matrix $DE$ satisfies (*). □

Note that the previous result cannot be generalized to rings with zero divisors, not even in dimensions $n = m = p = 1$. If there exist nonzero elements $d, e$ in $R$ with $de = 0$, then the $1 \times 1$ matrices $D = [e], E = [d]$ satisfy the hypothesis of Theorem 1 but matrix $DE = 0$ does not satisfy (*).

Our first main result is the existence of a Kalman decomposition for systems over Hermite rings $R$. If in addition $R$ is also an integral domain then the reduced form splits as a weakly reachable subsystem together with a system with no controls. The reduced form obtained in this way is the natural extension to multi-input systems of some well known result for single-input weakly reachable linear systems over Bézout domains.[21]

**Theorem 2** (A Kalman-like normal form)**.** *Let $R$ be a Hermite ring, and consider a system $(A, B)$ of size $(n, m)$ over $R$. Then:*

*(i) There exists an invertible matrix P such that the system* $(\hat{A} = PAP^{-1}, \hat{B} = PB)$ *has a Kalman decomposition*

$$\hat{A} = \begin{bmatrix} \hat{A}_1 & * \\ 0 & \hat{A}_0 \end{bmatrix}, \ \hat{B} = \begin{bmatrix} \hat{B}_1 \\ 0 \end{bmatrix}.$$

*Moreover,* $\hat{A}_1, \hat{B}_1$ *are of the form:*

$$\hat{A}_1 = \begin{bmatrix} A_{11} & * & \dots & \dots & * \\ D_2 & A_{22} & * & \dots & * \\ 0 & D_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \dots & 0 & D_k & A_{kk} \end{bmatrix}, \hat{B}_1 = \begin{bmatrix} D_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \tag{1}$$

*where for each* $i = 1, \dots, k$, $D_i \in R^{r_i \times r_{i-1}}$ *is in full row rank echelon form and* $A_{ii} \in R^{r_i \times r_i}$, *with* $m = r_0 \geq r_1 \geq \dots \geq r_k$.

*(ii) If R is also an integral domain, then the system* $(\hat{A}_1, \hat{B}_1)$ *is weakly reachable, and the integers* $r_i$ *are precisely the controllability (Kronecker) indices of the reachable system* $(\hat{A}_1, \hat{B}_1)$ *over the field of quotients of R*

$$r_1 = rank[B], \ r_2 = rank[B|AB] - rank[B],$$

*in general*

$$r_i = rank[B|AB| \cdots |A^{i-1}B] - rank[B|AB| \cdots |A^{i-2}B], \ for \ i = 2, \cdots, k.$$

*Proof.* (i) The proof will be done by induction on $n$. In the case $n = 1$, there is nothing to prove: Either $B = 0$ and $A$ is already in the desired block form with $\hat{A}_0$ filling whole of $A$, or $B \neq 0$ and $\hat{A}_1$ is all of $A$.

Assume $n > 1$. If $B = 0$, again there is nothing to prove. If $B \neq 0$, since $R$ is Hermite, there exists an invertible matrix $P_1$ such that $P_1 B = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}$, with $D_1 \in R^{r_1 \times m}$ in full rank row echelon form (in particular $r_1 \leq m$). Consider $P_1 A P_1^{-1}$ partitioned as $\begin{bmatrix} A_{11} & * \\ B' & A' \end{bmatrix}$, with $A_{11} \in R^{r_1 \times r_1}$, and the remaining blocks of appropriate sizes.

At this point, we can apply the induction hypothesis to the system $(A', B') \in R^{(n-r_1)\times(n-r_1)} \times R^{(n-r_1)\times r_1}$ and choose an appropriate invertible matrix $P'$ such that $(P'A'P'^{-1}, P'B')$ has the desired normal form:

$$P'A'P'^{-1} = \left[\begin{array}{ccccc|c} A_{22} & * & \cdots & \cdots & * & \\ D_3 & A_{33} & * & \cdots & * & \\ 0 & \ddots & \ddots & \ddots & \vdots & * \\ \vdots & \ddots & \ddots & \ddots & * & \\ 0 & \cdots & 0 & D_k & A_{kk} & \\ \hline & & 0 & & & \hat{A}'_0 \end{array}\right], P'B' = \left[\begin{array}{c} D_2 \\ 0 \\ \vdots \\ \vdots \\ 0 \\ \hline 0 \end{array}\right], \tag{2}$$

with $D_2, \dots, D_k$ full rank and in row echelon form. Now, defining the $n \times n$ invertible matrix $P = \begin{bmatrix} I_{r_1} & 0 \\ 0 & P' \end{bmatrix} P_1$, one has

$$PAP^{-1} = \begin{bmatrix} A_{11} & * \\ P'B' & P'A'P'^{-1} \end{bmatrix}, PB = \begin{bmatrix} D_1 \\ 0 \end{bmatrix}, \tag{3}$$

where the blocks $*$ may have changed, but this does not affect the procedure. In any case, combining Equation (3) with Equation (2) yields

$$PAP^{-1}=\left[\begin{array}{c|c} \overbrace{\begin{matrix} A_{11} & * & \cdots & \cdots & * \\ D_2 & A_{22} & * & \cdots & * \\ 0 & D_3 & \ddots & \ddots & \vdots \\ \vdots & \ddots & \ddots & \ddots & * \\ 0 & \cdots & 0 & D_k & A_{kk} \end{matrix}}^{\hat{A}_1} & * \\ \hline 0 & \hat{A}'_0 \end{array}\right], PB=\left[\begin{array}{c} \overbrace{\begin{matrix} D_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{matrix}}^{\hat{B}_1} \\ \hline 0 \end{array}\right],$$

which has the desired block form. Note that the integers $r_i$ (=number of rows of $D_i$) satisfy the order restrictions, because $r_0 = m$ by convention, $r_0 \geq r_1$ as was seen at the beginning of the proof, and $r_1 \geq \dots \geq r_k$ by the induction hypothesis.

(ii) Now assume that $R$ is a Bézout domain and observe that

$$\hat{B}_1=\begin{bmatrix} D_1 \\ 0 \\ \vdots \\ \vdots \\ 0 \end{bmatrix}, \hat{A}_1\hat{B}_1=\begin{bmatrix} * \\ D_2D_1 \\ 0 \\ \vdots \\ 0 \end{bmatrix}, \dots , {\hat{A}_1}^{k-1}\hat{B}_1=\begin{bmatrix} * \\ \vdots \\ \vdots \\ * \\ D_k \dots D_2D_1 \end{bmatrix}. \tag{4}$$

The above blocks could be only an extract of ${\hat{A}_1}^*\hat{B}_1$, the reachability matrix of $(\hat{A}_1, \hat{B}_1)$, but as will be seen, there are enough columns to select an appropriate minor of maximal rank, which will ensure that $(\hat{A}_1, \hat{B}_1)$ is weakly reachable. Indeed, since $D_1$ is in full row rank echelon form, in particular it satisfies the property (*) introduced before Theorem 1. Applying repeatedly Theorem shows that all blocks $D_i \dots D_2D_1$ have the property (*), for $i = 1, \dots , k$. Consequently, an appropriate selection of columns of $D_i \dots D_2D_1$, which we denote as $D'_i$, is a square lower triangular matrix with nonzero elements in the main diagonal. Precisely selecting these columns in the block ${\hat{A}_1}^{i-1}\hat{B}_1$ and putting all the blocks together gives the following selection of columns taken from ${\hat{A}_1}^*\hat{B}_1$:

$$\begin{bmatrix} D'_1 & * & \dots & * \\ 0 & D'_2 & \ddots & \vdots \\ \vdots & \ddots & \ddots & * \\ 0 & \dots & 0 & D'_k \end{bmatrix}. \tag{5}$$

This block is square and lower triangular with nonzero diagonal, therefore it has nonzero determinant, since $R$ has no zero divisors. Thus, we have found a nonzero minor of maximal size in ${\hat{A}_1}^*\hat{B}_1$, proving that $(\hat{A}_1, \hat{B}_1)$ is weakly reachable.

In order to prove the second part of (ii), it suffices to show that

$$\text{rank}[B|AB| \dots |A^{i-1}B] = r_1 + \cdots + r_i.$$

for each $i = 1, \dots , k$. Firstly observe that an immediate consequence of the Kalman decomposition is that the products $A^iB$ only depend on $\hat{A}_1$ and $\hat{B}_1$. To be concise, for all $i$ one has

$$[B|AB|\cdots|A^{i-1}B]=\left[\begin{array}{cccc} \hat{B}_1 & \hat{A}_1\hat{B}_1 & \cdots & {\hat{A}_1}^{i-1}\hat{B}_1 \\ \hline 0 & 0 & \cdots & 0 \end{array}\right].$$

Consequently, by a similar reasoning as above (extracting appropriate blocks $D'_i$) it follows that $rank[\hat{B}_1|\hat{A}_1\hat{B}_1| \dots |{\hat{A}_1}^{i-1}\hat{B}_1] = r_1 + \cdots + r_i$. This completes the proof. □

*Remark* 4 (A example over $\mathbb{Z}$)**.** Consider the following 2-input, 5-dimensional integer linear system $(A, B)$ given by

$$A = \begin{bmatrix} -9 & 12 & 6 & -12 & 6 \\ -20 & 23 & 13 & -24 & 12 \\ -23 & 28 & 15 & -28 & 14 \\ 11 & 4 & -5 & 4 & 2 \\ 57 & -38 & -34 & 53 & -20 \end{bmatrix}, \quad B = \begin{bmatrix} 1 & 2 \\ 1 & 7 \\ 1 & 7 \\ -1 & 3 \\ -3 & -11 \end{bmatrix}$$

and let us perform the procedure in order to obtain a reduced form

1. First obtain a row-echelon form equivalent to $B$:

$$P \cdot B = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 1 & 2 \\ 1 & 7 \\ 1 & 7 \\ -1 & 3 \\ -3 & -11 \end{bmatrix} = \begin{bmatrix} 1 & 2 \\ 0 & 5 \\ 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mapsto B$$

   operate state matrix:

$$P \cdot A \cdot P^{-1} = \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ -1 & 1 & 0 & 0 & 0 \\ 0 & -1 & 1 & 0 & 0 \\ 2 & -1 & 0 & 1 & 0 \\ 2 & 1 & 0 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} -9 & 12 & 6 & -12 & 6 \\ -20 & 23 & 13 & -24 & 12 \\ -23 & 28 & 15 & -28 & 14 \\ 11 & 4 & -5 & 4 & 2 \\ 57 & -38 & -34 & 53 & -20 \end{bmatrix} \cdot \begin{bmatrix} 1 & 0 & 0 & 0 & 0 \\ 1 & 1 & 0 & 0 & 0 \\ 1 & 1 & 1 & 0 & 0 \\ -1 & 1 & 0 & 1 & 0 \\ -3 & -1 & 0 & 0 & 1 \end{bmatrix} =$$

$$= \begin{bmatrix} 3 & 0 & 6 & -12 & 6 \\ 1 & 0 & 7 & -12 & 6 \\ 2 & 1 & 2 & -4 & 2 \\ 2 & 1 & -6 & 4 & 2 \\ 2 & 1 & -9 & 5 & 4 \end{bmatrix} \mapsto A$$

   Perform partition on $A$ according to structure of $B$ to obtain

$$\left[\begin{array}{cc|ccc} 3 & 0 & 6 & -12 & 6 \\ 1 & 0 & 7 & -12 & 6 \\ \hline 2 & 1 & 2 & -4 & 2 \\ 2 & 1 & -6 & 4 & 2 \\ 2 & 1 & -9 & 5 & 4 \end{array}\right], \left[\begin{array}{cc} 1 & 2 \\ 0 & 5 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right]$$

2. Consider the subsystem

$$(A_1, B_1) = \begin{bmatrix} 2 & -4 & 2 \\ -6 & 4 & 2 \\ -9 & 5 & 4 \end{bmatrix}, \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix}$$

   Reduce $B_1$ to a row-echelon form

$$P_1 \cdot B_1 = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \cdot \begin{bmatrix} 2 & 1 \\ 2 & 1 \\ 2 & 1 \end{bmatrix} = \begin{bmatrix} 2 & 1 \\ 0 & 0 \\ 0 & 0 \end{bmatrix} \mapsto B_1$$

   and operate the state-matrix of subsystem

$$P_1 A_1 P_1^{-1} = \begin{bmatrix} 1 & 0 & 0 \\ -1 & 1 & 0 \\ -1 & 0 & 1 \end{bmatrix} \begin{bmatrix} 2 & -4 & 2 \\ -6 & 4 & 2 \\ -9 & 5 & 4 \end{bmatrix} \begin{bmatrix} 1 & 0 & 0 \\ 1 & 1 & 0 \\ 1 & 0 & 1 \end{bmatrix} = \begin{bmatrix} 0 & -4 & 2 \\ 0 & 8 & 0 \\ 0 & 9 & 2 \end{bmatrix} \mapsto A_1$$

   then we can partition of $A_1$ according to structure of $B_1$ to obtain

$$\left[\begin{array}{c|cc} 0 & -4 & 2 \\ \hline 0 & 8 & 0 \\ 0 & 9 & 2 \end{array}\right]$$

3. Consider new subsystem

$$(A_2, B_2) = \begin{bmatrix} 8 & 0 \\ 9 & 2 \end{bmatrix}, \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

The procedure halts because of we have obtained a zero-input subsystem. Thus, the reduced form is as follows:

$$\left[\begin{array}{cc|c|cc} 3 & 0 & 0 & -12 & 6 \\ 1 & 0 & 1 & -12 & 6 \\ \hline 2 & 1 & 0 & -4 & 2 \\ \hline 0 & 0 & 0 & 8 & 0 \\ 0 & 0 & 0 & 9 & 2 \end{array}\right], \left[\begin{array}{cc} 1 & 2 \\ 0 & 5 \\ \hline 0 & 0 \\ 0 & 0 \\ 0 & 0 \end{array}\right] = (\hat{A}, \hat{B})$$

*Remark* 5 (A note on the regulator problem)**.** Regulation of a linear system with outputs

$$\begin{array}{rcl} x(t+1) & = & Ax(t) + Bu(t) \\ y(t) & = & Cx(t) \end{array}$$

consists of designing an observer $\xi(t)$, a feedback action $K$, and an output rejection $L$ to control the arising dynamic compensator.[15,22] Since the characteristic polynomial of dynamic compensator is $\chi(A + BF) \cdot \chi(A + LC)$, it follows that regulation problem reduces to the solution of pole shifting problem for pairs $(A, B)$ and $(A^t, C^t)$. In the case of systems over fields both pairs of matrices are usually reachable (i.e., the system with outputs is canonical), but this nice property is rarely given over commutative rings. The results in this paper would help in order to obtain the properties of dynamic compensators over a given linear system with outputs.

## 4 | COMPUTING ASSIGNABLE POLYNOMIALS

The reduced form over Hermite rings yields the detection of an invariant polynomial dividing every assignable polynomial to a given system:

**Theorem 3.** *Let $R$ be a Hermite ring, and consider a linear system $(A, B)$ over $R$. Then all assignable polynomials to $(A, B)$ are multiples of a monic polynomial which is an invariant associated to the system.*

*Proof.* Since an equivalence $(A, B) \cong (PAP^{-1}, PB)$ preserves the set of assignable polynomials, it can be assumed that $(A, B)$ has the normal form of theorem 2

$$A = \begin{bmatrix} A_1 & * \\ 0 & A_0 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix}$$

Since $A$ is upper triangular block matrix it follows that $A + BF = \begin{bmatrix} A_1 + B_1F_1 & * \\ 0 & A_0 \end{bmatrix}$; that is to say, for any feedback matrix $F = [F_1, *]$ the characteristic polynomial of $A + BF$ is $\chi(A + BF) = \chi(A_1 + B_1F_1)\chi(A_0)$. Therefore $\chi(A_0)$ always divides $\chi(A + BF)$ no matter the feedback matrix $F$, as desired. □

Because of $\chi(A_0)$ is a monic polynomial, Theorem 3 yields that multiplication by $\chi(A_0)$ maps the set $\mathrm{Pols}(A_1, B_1)$ into set $\mathrm{Pols}(A, B)$, and that mapping is a bijection.

*Remark* 6. Consider $R$ as a Bézout domain within the conditions of Theorem 3. This yields the subsystem $(A_1, B_1)$ is weakly reachable. Therefore the computation of the set $Pols(A, B)$ is reduced to the case of weakly reachable systems.

On the other hand, above invariant $\chi(A_0)$ can be obtained from the determinantal ideals of pencil of matrices $[B|z\mathbf{I} - A]$. Therefore, it is not necessary to compute the reduced form of Theorem 2 in order to get the invariant. The result is stated as follows.

**Theorem 4.** *Let $R$ be a Bézout domain. With the above notations, the ideal generated by $\chi(\hat{A}_0)$ in $R[z]$ is the smallest principal ideal of $R[z]$ containing $\mathcal{U}_n(B|z\mathbf{I} - A)$, the ideal generated by all the $n \times n$ minors of $(B|z\mathbf{I} - A)$.*

*Proof.* Let $H = [-B|zI - A]$. Consider any $n \times n$ invertible matrix $P$ and define $A' = PAP^{-1}$, $B' = PB$. Then

$$\underbrace{P[-B|z\mathbf{I} - A]}_{H}\begin{bmatrix} I & 0 \\ 0 & P^{-1} \end{bmatrix} = \underbrace{[-PB|zI - PAP^{-1}]}_{H'}.$$

This means that when we replace $(A, B)$ by $(A', B')$, the pencil $H$ is replaced by an equivalent pencil $H'$ (up to left and right multiplication by invertible matrices). In particular, one has that $\mathcal{U}_n(H) = \mathcal{U}_n(H')$. As a consequence, when computing $\mathcal{U}_n(H)$ one can assume that $(A, B)$ is already in the normal form of Theorem 2 and, taking into account the special form of pencil $H$,

$$A = \begin{bmatrix} A_1 & * \\ 0 & A_0 \end{bmatrix}, B = \begin{bmatrix} B_1 \\ 0 \end{bmatrix} \Rightarrow H = \begin{bmatrix} -B_1 & zI - A_1 & * \\ 0 & 0 & zI - A_0 \end{bmatrix}$$

Each $n \times n$ minor of $H$ is either 0 (if not all the columns of $zI - A_0$ are selected) or it factorizes as $\chi(A_0)$ multiplied by a maximal minor of $[-B_1|zI - A_1]$. Thus, we have proved that $\mathcal{U}_n(H)$ is contained in the principal ideal $(\chi(A_0))$.

In order to check the reverse inclusion, suppose that there exists a polynomial $q(z)$ such that $\mathcal{U}_n(H) \subseteq (q(z))$, we will prove that $(\chi(A_0)) \subseteq (q(z))$.

Consider two specific $n \times n$ minors of $H$. First, selecting the last $n$ columns one obtains an element $\chi(A_1)\chi(A_0)$ which belongs to $\mathcal{U}_n(H)$, so it must be of the form $q(z)\lambda(z)$ for some polynomial $\lambda(z)$. Because $\chi(A_1)$ and $\chi(A_0)$ are monic and $R$ is an integral domain, $q(z)$ must also be monic.

Next, assume that $(A_1, B_1)$ is weakly reachable and decomposed as in Theorem 2, with blocks $D_1, D_2, \cdots, D_k$ of full rank and in row echelon form. With the notations of Theorem 2, $A_1$ and $B_1$ have $r$ rows, and $r = r_1 + \cdots + r_k$. Now, we select precisely the $r$ columns of $[-B|zI - A_1]$ which contain all the pivots of $D_1, \cdots, D_k$. Since the blocks $D_i$ have no intersection with the main diagonal, the indeterminate $z$ will not appear, therefore this second $n \times n$ minor will be of the form

$$d'_1 \cdots d'_r \chi(A_0) = q(z)\mu(z)$$

for certain elements $d'_i \neq 0$ in $R$, and for some polynomial $\mu(z)$. But $\chi(A_0)$ and $q(z)$ are monic, which means that the nonzero element $d = d'_1 \cdots d'_r$ must divide $\mu(z)$, so $\chi(A_0)$ is a multiple of $q(z)$, and the proof is complete. □

*Remark* 7. The following example shows that the above theorem cannot be generalized to Hermite rings with zero divisors.

Let $R = \mathbb{Z}/4\mathbb{Z}$ and consider the system $(A, B)$ given by

$$A = \begin{bmatrix} 0 & 0 \\ 0 & 0 \end{bmatrix}, B = \begin{bmatrix} 2 \\ 0 \end{bmatrix}$$

With the previous notations, $(A, B)$ is in the block form of Theorem 2, with $A_0 = [0]$ and $\chi(A_0) = z$.

Also, we have $H = [-B|zI - A] = \begin{bmatrix} 2 & z & 0 \\ 0 & 0 & z \end{bmatrix}$, so $\mathcal{U}_2(H)$ is spanned by the polynomials $(z^2, 2z)$. But the equalities

$$(z + 2)2 = 2z, \qquad (z + 2)^2 = z^2$$

yield that the principal ideal $(z + 2)$ contains $\mathcal{U}_2(H)$, while it does not contain $(\chi(A_0)) = (z)$.

*Remark* 8. We would like to note here that similar results were used[3] to study (a) coefficient asisignability, which is the study of when is every monic polynomial feedback assignable to a reachable system; and (b) strong coefficient assignability, which is the study of when every monic polynomial of degree equal to the residual rank of a system divides some assignable polynomial. Our approach is rather different because we focus on row-echelon reduction, while strong coefficient assignability focus on residual rank. The account of feedback assignable polynomials by means of reduction would be of interest for systems with small residual rank. Next we give a concrete example.

Consider the ring of finite hyperreals[7] $\mathbb{R}_0$. This is a local domain, in fact a valuation ring, and hence Bézout. Therefore $\mathbb{R}_0$ is a strong CA-ring.[17] The maximal ideal of $\mathbb{R}_0$ is the set of infinitesimals, $I$, and the residual field is $\mathbb{R}_0/I = \mathbb{R}$.

Change of scalars $st : \mathbb{R}_0/I = \mathbb{R} \to \mathbb{R}$ is usually called standard part. Hence the residual rank of a system over $\mathbb{R}_0$ is just the rank of the associated standard system over $\mathbb{R}$.

Since $\mathbb{R}_0$ is a Bézout domain it follows that the rank of a matrix $M$ is just the number of nonzero rows in a row-echelon form of $M$. Because of the field of quotients of $\mathbb{R}_0$ is the field of hyperreals $\mathbb{R}^*$, it follows that rank($M$) equals the rank of extended matrix $M \otimes_{\mathbb{R}_0} \mathbb{R}^*$.

As a numerical example, consider an infinitesimal $\varepsilon \in I$, and the linear system over $\mathbb{R}_0$ given by

$$\Sigma = \begin{bmatrix} 2 & (1-\varepsilon) & \varepsilon \\ 1 & 1 & 1 \\ 4 & -1 & 0 \end{bmatrix}, \begin{bmatrix} \varepsilon & \varepsilon^2 \\ \varepsilon & \varepsilon^2 \\ \varepsilon & \varepsilon^2 \end{bmatrix}$$

Because of st($B$) = (0) it follows that the residual rank of above system is 0. Hence, strong coefficient assignability gives no information about the set of assignable polynomials.

On the other hand, $\varepsilon$ is not zero and hence a unit in $\mathbb{R}^*$. Hence, the rank of matrix $B$ equals 1 in $\mathbb{R}^*$. Then reduced form should give some information.

In fact, a reduced form for $\Sigma$ is the following:

$$\left[\begin{array}{c|cc} 7 & (1-\varepsilon) & \varepsilon \\ \hline 0 & \varepsilon & (1-\varepsilon) \\ 0 & (-2+\varepsilon) & -\varepsilon \end{array}\right], \left[\begin{array}{cc} \varepsilon & \varepsilon^2 \\ \hline 0 & 0 \\ 0 & 0 \end{array}\right]$$

Therefore, by Theorem 3, every assignable polynomial is a monic multiple of $z^2 + (2 - 3\varepsilon)$. Moreover, the structure of system yields that we are able to compute exactly its set of assignable polynomials as follows

$$\text{Pols}(\Sigma) = \left\{ [z^2 + (2 - 3\varepsilon)](z - (7 + \varepsilon\lambda)) : \lambda \in \mathbb{R}_0 \right\}$$

## 5 | CONCLUSIONS

A reduced form for linear systems over Hermite rings is given. This reduced form is inspired in Kalman's reduced form for linear control systems over $\mathbb{C}$. The reduced form gives information about what kind of polynomials can be assigned to a given system. In particular, all assignable polynomials are multiples of a monic polynomial which is an invariant of the system. This invariant polynomial can be computed from the determinantal ideals of the pencil associated to the system in the case of $R$ being a Bézout domain. A concrete example of computation of reduced form over $\mathbb{Z}$ is given by means of repeated reduction of block matrices to a triangular form. Finally we consider an example of computation of reduced form of a residual rank zero linear system. Hence it is possible to obtain its set of assignable polynomials from a reduced form even in case of residual rank zero. This is an improvement over classical results on strong coefficient assignability property.

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## CONFLICT OF INTEREST

This work does not have any conflicts of interest.

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