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Qualified voting mechanisms

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QUALIFIED VOTING MECHANISMS*

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Abstract

We study voting mechanisms, which consist of two elements: a profile of sets of votes (this profile describes the votes that voters are allowed to cast) and a voting scheme (which explains how to aggregate those votes). To investigate how these two elements interact, we impose some properties on the sets of votes (i.e., *regularity*) and on the voting scheme (i.e., *candidate monotonicity*, *candidate anonymity*, and *weak neutrality*). We characterize the family of voting schemes that satisfy some of those properties and analyze the role played by the structure of the sets of votes in these characterizations.

Keywords: Voting, approval voting rule, plurality rule, monotonicity, anonymity, neutrality, qualified voting.

JEL Classification: D7

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1 Introduction

Most of the decisions that a society has to make involve selecting a representative, e.g., electing the prime minister or congressmen, the head of a department, the person to hire for a particular task, or even whether a law has to be changed. In general, the problem consists of a group of voters who have to choose one from a set of potential candidates.

In essence, any voting situation is described by two key ingredients: the information we collect from the voters and how we aggregate this information to choose a winner. The way in which both elements are combined, as well as the properties they satisfy, determine the particular features of the voting processes. Regarding the first ingredient, each voter has a *set of votes* that describes the possible votes she can cast. This set may have a different structure, depending on the situation we need to accommodate. For example, voters are sometimes limited to vote for only one candidate, but in others cases voters are allowed to support as many candidates as they wish; similarly, there are cases where voters can abstain and cases where they are obliged to vote. Each voter casts a vote from her set of votes. A *voting configuration* is a list of votes, one vote for each voter. The second ingredient describes how to choose a candidate taking as input a voting configuration. We call this ingredient a *voting scheme.* There are different types of voting schemes, e.g., a dictatorial voting scheme where the dictator alone determines the winning candidate, the voting scheme that selects the candidate who garners the most votes, or we can choose a candidate only if the number of votes she obtains is above a certain threshold. A *voting mechanism* is a pair: a profile of sets of votes (one for each voter) and a voting scheme. In this paper, we study voting mechanisms and analyze the interactions between the structure of the sets of votes and the properties that the voting schemes may satisfy.

The appeal of voting mechanisms comes from the properties they satisfy, which should be natural requirements for both the sets of votes and the voting schemes. We focus on four properties of voting schemes. *Candidate monotonicity* states that if a candidate is chosen and her support subsequently increases while the support for her opponents decreases, then this candidate should remain chosen.[1] *Strong candidate monotonicity* is a more demanding requirement: it says that if a candidate is selected and her support increases, then the voting scheme must select either the same candidate or some other candidate whose support has also increased. *Candidate anonymity* states that the election of the winner does not depend on who votes for whom. Finally, *weak neutrality* requires that, whenever possible, candidates' identities are not determinant. Regarding the set of votes, we say that it is *regular* essentially when voters are not limited by the names of the candidates for whom

[1]A similar definition of candidate monotonicity appears in Barbera et al. (1991)

they can cast votes.

To find the voting mechanisms for which the sets of votes and voting schemes that may satisfy the above properties, we define the *qualified voting schemes*. They constitute a family, each of whose elements is determined by a *weights matrix* that has as many rows as candidates, and as many columns as voters plus one. The entries of the matrix must satisfy two conditions. First, they must be distinct from each other, and second, their rows must be increasing. To each weights matrix, we associate a qualified voting scheme, which functions as follows: form all the pairs candidate/number-of-voters-supporting-her. Each pair corresponds to an entry in the matrix, and we choose the candidate whose corresponding entry is the largest. It is quite obvious that different weights matrices lead to different qualified voting schemes. Some of the best well-known voting rules in the literature are particular cases of our qualified voting mechanisms. Hence, our family contains the approval voting rule that was introduced in Brams and Fishburn (1978) (i.e., voters can vote for all of the candidates they approve, and the winner is the candidate who receives the most support),[2] the plurality rule (i.e., voters are limited to supporting at most one candidate, and the candidate obtaining the most votes is chosen), the weighted approval voting rules defined by Massó and Vorsatz (2008) (i.e., which generalize the approval voting rule by introducing asymmetries on the candidates), among others.

Usually, suitable combinations of properties characterize one or several voting schemes. Our main result shows that when the sets of votes are regular, the qualified voting schemes are the unique voting schemes that satisfy both candidate monotonicity and candidate anonymity. We further prove that if we also impose weak neutrality, then we end up with voting schemes that follow the spirit of the approval and plurality voting rules. As we have already mentioned, the structure of the sets of votes and the properties that characterize the voting schemes are closely related. If we relax the regularity requirement, these results are not longer true. However, the qualified voting schemes still emerge when we replace candidate monotonicity with strong candidate monotonicity, even when the set of votes is not regular.

The structure of the paper is as follows. In Section 2, we introduce the basic setup and illustrate some examples of voting mechanisms. In Section 3, we present the properties of the voting schemes that we study in this paper. In Section 4, we define the qualified voting schemes, and we show that they generalize some of the voting rules that are found in the literature. In Section 5, we present our main results. Finally, in Section 6, we conclude with some final remarks.

[2]See Brams and Fishburn (2007) and Laslier and Sanver (2010) for two surveys on approval voting.

2 Voting mechanisms.

Let $N = \{1, \ldots, n\}$ be the **set of voters** who must select a candidate, let $K = \{1, \ldots, k\}$ $(k > 2)$ be the **set of candidates** from whom voters must choose, and let 2^K be the set of all possible non-empty subsets of K. Elements of N are denoted by $i, j, \ldots$, and elements of K are denoted by $x, y, \ldots$. Subsets of 2^K are denoted by S and S', and subsets of N are denoted by M and M'.

For each voter $i \in N$, let $V_i \subseteq 2^K \cup \phi$ be the set of "admissible" votes for i.[3] This set specifies the votes that voter i can cast. We refer to element $v_i \in V_i$ as i's **vote**. A **voting configuration** is an specification of votes, one vote for each voter in N, $v_N = (v_1, \ldots, v_n) \in V_1 \times \cdots \times V_n = V^N$.

Example 2.1. The sets of votes $V_1, \ldots, V_n$ may have different structures, depending on the situation we need to accommodate.

- If $V_i = 2^K \cup \phi$ for all $i \in N$, then each voter simply submits the names of the candidates she approves (if there are any).
- A minor modification of the previous case is when $V_i = 2^K$ for all $i \in N$. In this new case, the voters are obliged to vote for at least one candidate and they cannot abstains. On the surface, this may appear to be an insignificant change, but, as we show later, it is significant.
- In other situations, such as the election of a legislature, voters are limited to voting for no more than one candidate, that is, $V_i = K \cup \phi$ for all $i \in N$.
- A compromise between the previous cases is when voters are limited in the number of candidates they can support. For example, $V_i = \{S \subseteq K : \#S \leq 3\}$ for all $i \in N$.
- Finally, the Papal Conclave to elect the pope is a particular situation where each of the cardinals can vote for any member of the College of Cardinals except himself. In this case, the sets of candidates and voters are the same; thus, $K = N$ and $V_i = N \setminus \{i\}$ for all $i \in N$.

For each candidate $x \in K$ and each voting configuration $v_N \in V^N$, let $M_x(v_N) = \{i \in N : x \in v_i\}$ be the **support for candidate** x, i.e., the set of voters who are voting for x in voting configuration v_N. We denote by $m_x(v_N)$ the cardinality of $M_x(v_N)$, $m_x(v_N) = \#M_x(v_N)$.

Let $\alpha \in \mathbb{Z}_+$, we define $V^\alpha = \{S \in 2^K \cup \phi : \#S = \alpha\}$. Hence, V^0 contains the votes in which voters are limited to supporting no candidate ($V^0 = \phi$), V^1 contains the

[3] To simplify the notation, we write $A \cup \phi$ instead of $A \cup \{\phi\}$.

votes in which voters are limited to supporting exactly one candidate ($V^1 = K$). We say that a set of votes V is *regular* when (i) votes are not restricted by the identity of the candidates (e.g., if $V \cap V^1 \neq \phi$ then $V^1 \subseteq V$), and (ii) if the voters are allowed to support α candidates, then they are also allowed to support any number of candidates smaller than α.

Regularity. If $V^\alpha \cap V \neq \phi$ then $V^\beta \subseteq V$ for all $\beta \leq \alpha$.

Among the sets of votes shown in Example 2.1, some are regular, but others are not. For example, $V = 2^K \cup \phi$ and $V = K \cup \phi$ are regular. In contrast, $V = K$ is not regular because it does not contains $V^0 = \phi$, and $V_i = K - \{i\}$ is not either because it distinguishes candidate i from the others, i.e., $V_i \cap V^1 \neq V^1$.

A **voting scheme** is a way to aggregate votes to choose a single candidate. A voting scheme is a mapping $g : V^N \longrightarrow K$ that for each $v_N \in V^N$ selects an element of K, $g(v_N) \in K$.

Given a voting scheme g, let A_g denote the range of g:

$$A_g = \{x \in K : g(v_N) = x \text{ for some } v_N \in V^N\}.$$

As with the sets of votes, we present some examples of voting schemes.

Example 2.2. Let σ be an ordering over the set of candidates K that serves as a tie-breaking rule (we need a tie-breaking rule because we can choose only one candidate). Given $S \subseteq K$, let us denote by $S(\sigma) \in S$ the candidate that σ ranks first among those within S.

- Dictatorship of voter $i \in N$, g^{dic_i}: voter i has the power to impose the candidate for whom she is voting (obviously, σ is needed to break the ties when $\#v_i \geq 2$). More specifically, $g^{\text{dic}_i}(v_N) = v_i(\sigma)$.

- Majority voting scheme, g^{maj}: the candidate who obtains the most votes wins the election. That is, $g^{\text{maj}}(v_N) = M(\sigma)$, where $M = \arg\max_{x \in K} m_x(v_N)$.

- Anti-majority voting scheme, $g^{\text{anti-maj}}$: the candidate who obtains the fewest votes wins the election, i.e., $g^{\text{anti-maj}}(v_N) = M(\sigma)$, where $M = \arg\min_{x \in K} m_x(v_N)$.

- Incumbency of candidate $x \in K$, g^{inc_x}: a candidate can potentially be chosen only if she receives more than $\frac{n}{2}$ votes. If necessary, we apply the tie-breaking rule σ. If no candidate reaches that threshold of support, then x is elected. In other words, x can be seen as a status quo, but that may change if there is enough support for an alternative. More specifically, if we define $M'(v_N) =$

$\{y \in K : m_y(v_N) > \frac{n}{2}\}$,

$$g^{\text{inc}_x}(v_N) = \begin{cases} x & \text{if } M'(v_N) = \phi \\ M'(v_N)(\sigma) & \text{otherwise} \end{cases}$$

A **voting mechanism** is a pair (V^N, g) where $V^N = V_1 \times \ldots \times V_n$ is a profile of sets of votes and g is a voting scheme.

Table 1 shows that different combinations of these two ingredients lead to different voting mechanisms. The rows contain the sets of votes, and the voting schemes are in the columns. Therefore, each entry in the table is a voting mechanism. Thus, if we stay in the same column, the advantages and disadvantages of a voting mechanism are mostly due to the direct or indirect impact of a change in the set of votes. Similarly, by keeping the row fixed, we can analyze the behavior of the voting mechanism when the corresponding voting scheme varies. If we combine $V_i = 2^K \cup \phi$ and $V_i = K \cup \phi$ with the majority voting schemes, we obtain the standard approval and plurality mechanisms, respectively. In both mechanisms, the winner is the candidate who gets the most support (or one of those, in the event of tie), but this mechanisms differ with respect to the votes that voters are allowed to cast. The Papal Conclave used the majority voting scheme to elect a new pope for several centuries. In this case, each cardinal could not vote for himself. The anti-approval and anti-plurality voting mechanisms are the counterparts of the approval and plurality mechanisms, respectively, because the winner is the candidate who receives the fewest votes. The lifeboat game also uses the anti-majority voting scheme: a ship with ten people (e.g., a teacher, a plumber, an economist, a physician,...) is wrecked at sea, and the unique lifeboat only has nine slots. Each passenger votes for the person, excluding himself, whom he thinks should survive. The passenger with the fewest votes stays on the ship and dies. Constitutional reforms are also examples of voting mechanisms. Let $V_i = \{A, C\} \cup \phi$, where ϕ indicates abstention, A indicates a vote in favor of amending the constitution, and C indicates a vote against amending the constitution. The voters are the congressmen. To pass the amendment, it is necessary to obtain the support of an absolute majority of the congress, and if that threshold is not reached, then the constitution remains unaltered. That is, C is the incumbent, and the incumbent only changes if the alternative A has enough support.

3 Properties

Not every mechanism is equally appealing. The sets of votes should be natural, simple and non-discriminatory, and the voting schemes should satisfy some desirable and reasonable properties. With this goal in mind, we introduce some appealing requirements that we impose on voting schemes.

Sets of votes	Voting schemes		
	g^{maj}	$g^{\text{anti-maj}}$	g^{inc_x}
$V_i = 2^K \cup \phi \ \forall i \in N$	Approval voting mechanism	Anti-approval voting mechanism	
$V_i = K \cup \phi \ \forall i \in N$	Plurality voting mechanism	Anti-plurality voting mechanism	Constitutional reforms
$V_i = K - \{i\} \ \forall i \in N$	Election of the pope from the 13th to early 17th centuries	Lifeboat game	

Table 1: Different voting schemes.

Suppose that candidate x is elected in some voting configuration, but then some voters change their votes such that the support for x increases while the support for the rest of the candidates decreases. Then x must still be elected in the new voting configuration. We refer to this property as *candidate monotonicity.*

Candidate monotonicity. For each $v_N, \overline{v}_N \in V^N$ such that $M_x(v_N) \subseteq M_x(\overline{v}_N)$ and $M_y(v_N) \supseteq M_y(\overline{v}_N)$ for all $y \in A_g - x$, if $g(v_N) = x$ then $g(\overline{v}_N) = x$.

Strong candidate monotonicity is a more demanding version of the previous principle. Assume that candidate x is elected in configuration v_N, and now consider another voting configuration $\overline{v}_N$ in which x, and potentially other candidates, have increased their support. *Strong candidate monotonicity* requires that, in the new voting configuration $\overline{v}_N$, we choose either x or any other candidate whose support has increased.

Strong candidate monotonicity. For each $v_N, \overline{v}_N \in V^N$ such that $M_x(v_N) \subseteq M_x(\overline{v}_N)$, it must hold that $g(\overline{v}_N) \in S(v_N, \overline{v}_N) \cup g(v_N)$, where $S(v_N, \overline{v}_N) = \{y \in A_g : M_y(v_N) \subset M_y(\overline{v}_N)\}$.

Remark 3.1. Let us consider that $N = \{1, 2, 3\}$, $K = \{x, y, z\}$, and $V_i = 2^K$ for all $i \in N$. Let $v_N = (\{x, y\}, \{y\}, \{z\})$ and $v'_N = (\{x, y\}, \{y\}, \{x\})$ be two voting configurations. Assume that $g(v_N) = x$. When we pass from v_N to v'_N, the candidate x increases her support, but no other candidate do it. If candidate monotonicity is applied it must happen that $g(v'_N) = x$. Consider the voting configuration $v''_N = (\{x, y\}, \{y\}, \{x, y\})$. When passing from v_N to v''_N, we observe that whereas x and y have obtain more support, z has received no vote. Candidate monotonicity does not impose any restriction in this situation, and furthermore, $g(v''_N) \in \{x, y, z\}$, which implies that even candidate z may win the election under v''_N. However, if we require g to satisfy strong candidate monotonicity instead of candidate monotonicity, then it must happen that $g(v''_N) \in \{x, y\}$.

The next property states that it is not relevant who votes for whom, so the election depends only on the number of votes obtained by each candidate.

Candidate anonymity. For each $v_N, \overline{v}_N \in V^N$ such that $m_x(v_N) = m_x(\overline{v}_N)$ for all $x \in K$, it must hold that $g(v_N) = g(\overline{v}_N)$.

Our last requirement captures the principle of *neutrality*, which is commonly imposed in the literature. It says that candidates should be treated equally and that we cannot discriminate among them based on their names. Because we must always select one candidate, this principle cannot be directly applied to our context. In a simple situation in which all the candidates are voted by all of the voters, the mere application of neutrality would not meet the requirement of selecting a single candidate. Thus, we need to adapt this principle to our framework. Given $x \in K$ and $v_N \in V^N$, let us define $D_x(v_N) = \{y \in K : M_x(v_N) \neq M_y(v_N)\}$ as the set of candidates whose supporters are distinct from the supporters of x. *Weak neutrality* requires that changing the labels within $D(v_N)$ does not alter the voting scheme. In other words, the voting scheme should be neutral when such a possibility exists.

Weak neutrality. For any $v_N, \overline{v}_N \in V^N$, and $y \in D_{g(v_N)}(v_N)$ such that $M_y(\overline{v}_N) = M_{g(v_N)}(v_N)$, $M_{g(v_N)}(\overline{v}_N) = M_y(v_N)$ and $M_x(\overline{v}_N) = M_x(v_N)$ for all $x \in K \setminus \{y, g(v_N)\}$, it must hold that $g(\overline{v}_N) = y$.

4 Qualified voting schemes

In Section 2, we showed some examples of voting schemes. In this section, we propose a new family, each of whose members is parametrized by a matrix. Later, we will relate this family with the aforementioned properties.

Let $\omega \in \mathbb{R}^k_+ \times \mathbb{R}^{n+1}_+$ be a weights matrix with k rows and $n+1$ columns. Each row corresponds to a candidate $x \in K$, and each column represents the number of voters who may be supporting the candidate on the row. A general entry in the weights matrix ω is denoted by $\omega_{x,m}$. Matrix ω must satisfy the following two conditions:

(a) $\omega_{x,m} < \omega_{x,m+1}$ $\forall m \in \{1, \ldots, n-1\}$ and $\forall x \in K$.

(b) $\omega_{x,m} \neq \omega_{y,m'}$ $\forall x, y \in K$ and $\forall m, m' \in [1, n+1]$

Condition (a) implies that the weights on the rows are increasing, and Condition (b) requires that all of the entries must be different from each other. We denote by Ω the collection of all such weights matrices.

Example 4.1. Let $N = \{1, 2, 3\}$ and $K = \{x, y, z\}$ be the set of voters and candidates, respectively. One element of Ω is

$$\begin{array}{cc} & \begin{array}{cccc} 0 & 1 & 2 & 3 \end{array} \\ \omega = \begin{array}{c} x \\ y \\ z \end{array} & \begin{pmatrix} 0.2 & 3 & 4 & 8 \\ 0.3 & 2 & 5 & 9 \\ 0.1 & 1 & 6 & 7 \end{pmatrix} \end{array}$$

The first row corresponds to candidate x, the second row corresponds to candidate y, and the third row corresponds to candidate z. The first column refers to the case in which a candidate $h \in K$ has no support, the second column refers to the case in which a candidate $h \in K$ is supported by just one voter, and so on. In this matrix, the entry $\omega_{y,2}$ equals to 5 and it applies when candidate y gets the support of two out of three voters. Below, we explain how to use the information in the weights matrix to construct voting schemes.

Given a weights matrix $\omega \in \Omega$, we define a *qualified voting scheme* as follows:

Qualified voting associated with $\boldsymbol{\omega}$, Q^ω. For each $v_N \in V^N$,

$$Q^\omega(v_N) = \arg\max_{x \in K} \omega_{x,m_x(v_N)+1}.$$

A qualified voting scheme selects the candidate whose corresponding pair candidate/support has the highest weight in the matrix ω. The next example shows how qualified voting schemes function:

Example 4.2. Let $N = \{1, 2, 3\}$ and $K = \{x, y\}$ be the sets of voters and candidates, respectively. Each candidate can be supported by 0, 1, 2, or 3 voters. Let ω be the following weights matrix:

$$\omega = \begin{pmatrix} 0 & 2 & 3 & 7 \\ 1 & 4 & 5 & 6 \end{pmatrix}$$

- If $v_N \in V^N$ is such that $v_N = (x, y, x)$, then the support sets would be $M_x(v_N) = \{1, 3\}$ and $M_y(v_N) = \{2\}$, and the cardinalities of these sets are $m_x(v_N) = 2$ and $m_y(v_N) = 1$. Because $\omega_{x,m_x(v_N)} = \omega_{x,2} = 3 < 4 = \omega_{y,1} = \omega_{y\,m_y(v_N)}$, we obtain that $Q^\omega(v_N) = y$.
- If $v_N \in V^N$ is such that $v_N = (\{x, y\}, \{x\}, \{y\})$, then $m_x(v_N) = 2$ and $m_y(v_N) = 2$. Because $\omega_{x,2} = 3 < 5 = \omega_{y,2}$, we obtain that $Q^\omega(v_N) = y$.

Different weights matrices can lead to the same qualified voting scheme. For example, let $\omega \in \Omega$, and let $\lambda\omega \in \Omega$ be the weights matrix all of whose entries have been multiplied by the scalar $\lambda \in \mathbb{R}_{++}$. Then $Q^\omega = Q^{\lambda\omega}$.

A **qualified voting mechanism** is a pair (V^N, Q^ω), where V^N is a profile of sets of votes and Q^ω is a qualified voting scheme. Next, we show that some of the voting mechanisms in the literature are particular cases of qualified voting mechanisms.

Let us consider the following weights matrix:

$$\overline{\omega} = \begin{array}{c} \\ x \\ y \\ z \\ \vdots \\ \\ \end{array} \begin{array}{c} \begin{array}{ccccc} 0 & 1 & 2 & \dots & n \end{array} \\ \begin{pmatrix} 0+0\varepsilon & 1+0\varepsilon & 2+0\varepsilon & \dots & n+0\varepsilon \\ 0+1\varepsilon & 1+1\varepsilon & 2+1\varepsilon & \dots & n+1\varepsilon \\ 0+2\varepsilon & 1+2\varepsilon & 2+2\varepsilon & \dots & n+2\varepsilon \\ \vdots & \vdots & \vdots & \ddots & \vdots \\ 0+(k-1)\varepsilon & 1+(k-1)\varepsilon & 2+(k-1)\varepsilon & \dots & n+(k-1)\varepsilon \end{pmatrix} \end{array},$$

where $\varepsilon < \frac{1}{k}$. We observe in $\overline{\omega}$ that any entry is larger than all of the entries of any previous column (for instance, $2+1\varepsilon$ is larger than each entry of the first two columns), and therefore, the qualified voting scheme associated with this weights matrix always chooses the winer among the candidates with the largest support. In other words, the $Q^{\overline{\omega}}$ coincides with the majority voting scheme. We denote by $\overline{\Omega}$ the set of all weights matrices that are like $\overline{\omega}$. As stated in Section 2, the usual approval voting mechanism is a voting mechanism in which the sets of votes are $V_i = 2^K \cup \phi$ for all $i \in N$ and the voting scheme is g^{maj}. Therefore, the approval voting mechanism is a particular case of the qualified voting mechanisms $((2^K \cup \phi)^N, Q^\omega)$ when $\omega \in \overline{\Omega}$. Another important advantage of the qualified voting schemes that we present is the possibility of breaking ties in a more complex way. Traditionally, in the approval voting mechanism, ties are settled using a priority ordering over the candidates. This ordering implies that, regardless to which number they tie, this tie is always broken in favor of the same candidate. Inasmuch as the qualified voting schemes are more general in this matter, we can break ties differently depending, for example, on whether the tie is odd or even.

Analogously, it is easy to check that the plurality voting mechanism is also included in the qualified voting family, where the sets of votes are $V_i = K \cup \phi$ for all $i \in N$ and the voting schemes are Q^ω with $\omega \in \overline{\Omega}$.

Massó and Vorsatz (2008) introduced what they called the *weighted approval voting rules* as generalizations of the approval voting rule. As defined, these rules may select select more than one candidate. For the sake of comparison with our framework, we can easily assume that there exists a tie-breaking rule (i.e., an ordering over the set of candidates K) that results in a single winner. In terms of voting mechanisms, the set of votes in Massó and Vorsatz (2008) are $V_i = 2^K \cup \phi$ for all $i \in N$, and the voting scheme works as follows: each candidate $x \in K$ has an associated weight $\alpha_x \in \mathbb{R}_+$ (different weights lead to different weighted approval voting schemes). For a given voting configuration $v_N \in V^M$, the weighted support of candidate x is $\alpha_x \cdot m_x(v_N)$. The candidate with the highest weighted support wins the election (again, a tie-breaking rule is applied if necessary). The voting mechanisms defined in Massó and Vorsatz (2008) also belong to the qualified voting family.

However, not every generalization of the approval voting rule can be seen as a qualified voting mechanism. For example, the proposals by Alcalde-Unzu and Vorsatz (2009) are not in the qualified voting family.

Let us consider the following weights matrix.

$$\omega' = \begin{array}{c} \\ x \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{ccccc} 0 & & \delta & & n \end{array} \\ \left(\begin{array}{c|ccc} \multicolumn{4}{c}{\omega^A} \\ \hline & \omega^C_{1,1} & \cdots & \omega^C_{1,n-\delta+1} \\ \omega^B & \vdots & \ddots & \vdots \\ & \omega^C_{k-1,1} & \cdots & \omega^C_{k-1,n-\delta+1} \end{array} \right) \end{array}$$

As one can observe, this matrix consists of three blocks. The first block, ω^A, corresponds to the row of candidate x. The second block, ω^B, contains all of the rows except for x, and all the columns associated with supports from 0 to $\delta - 1$. Finally, the third block, ω^C, consists of the rest of the matrix, i.e., all of the rows but x, and the columns associated with supports from δ to n. Subject to the conditions of being a weights matrix in Ω, let us assume that (a) any entry of the block ω^A is larger than any other entry of the block ω^B, and (b) any entry of the block ω^C is larger than any other entry of the block ω^A. For example,

$$\omega' = \begin{array}{c} \\ x \\ \\ \\ \\ \end{array}\begin{array}{c} \begin{array}{cccccc} 0 & & & 3 & & 5 \end{array} \\ \left(\begin{array}{ccc|ccc} 10 & 11 & 12 & 13 & 14 & 15 \\ \hline 1 & 4 & 7 & 16 & 21 & 23 \\ 2 & 5 & 8 & 17 & 20 & 22 \\ 3 & 6 & 9 & 18 & 19 & 24 \end{array} \right) \end{array}$$

By construction, the $Q^{\omega'}$ voting scheme selects x when none of the other candidates obtain the support of a majority of the voters (3 out of 5). If one or several candidates pass that threshold, then one of them is elected, depending on the candidate's weights and supports. In other words, the $Q^{\omega'}$ voting scheme coincides with the g^{inc_x} voting scheme that is presented in Example 2.2. Therefore, given a fixed $x \in K$, the mechanisms $(V^N, g^{\mathrm{inc}_x})$ are particular cases of the qualified voting mechanisms. Of course, generalizations can be made by varying the threshold to change the status quo; in fact, such a threshold could be different for different candidates.

5 Results

In this section, we present our main results. The first two theorems state that, as long as the sets of votes $V_1 \dots, V_n$ are regular, some suitable combinations of properties characterize either the qualified voting schemes or the majority schemes, depending on the properties we impose. In addition, we also show that the voting mechanism, which is understood as the paired sets of votes with their voting schemes, must be taken as a whole because the assumption about the regularity of the V_is plays a crucial role in these characterizations.

Theorem 5.1. *Let (V^N, g) be a voting mechanism such that V_i is regular for all $i \in N$. Then, g satisfies both candidate anonymity and candidate monotonicity if and only if g is a qualified voting scheme.*

Proof. It is not difficult to verify that Q^ω satisfies the properties in the statement of the theorem. To see the converse, let g be a voting scheme that is not in the Q family but fulfills the requirements of both candidate anonymity and candidate monotonicity. First, we use the information on g to obtain a weights matrix ω. Second, we show that Q^ω coincides with g.

We can define the function $G : \mathbb{R}^k \longrightarrow K$ that provides the elected candidate taking as input a vector of supports $(m_1, \dots, m_k)$.

$G(m_1, \dots, m_k) = x \Leftrightarrow g(v_N) = x$ for some $v_N \in V^N$such that $m_y(v_N) = m_y \ \forall y \in K$

The function G is well defined because g is *candidate anonymous*, which implies that G does not depend on the identity of the voters in the sets $M_x(v_N)$, but it does depend on their cardinalities $m_x(v_N)$.

Now, we define the weights matrix $\omega \in \mathbb{R}^k \times \mathbb{R}^{n+1}$ as a matrix that solves the following system of inequalities:

(i) $\omega_{x,i} \leq \omega_{x,(i+1)} \ \forall i \in \{1, .., n-1\}$ and $\forall x \in K$.

(ii) $G(m_1, \dots, m_k) = x$ if and only if $\omega_{x,m_x} > \omega_{y,m_y} \ \forall y \in K - \{x\}$.

To show that the previous system has a solution, it is enough to prove that the system does not generate cycles. We proceed by contradiction. Assume that the system does generate cycles. In that case, we obtain (directly or indirectly) a situation where $\omega_{x,u} < \omega_{y,v} < \omega_{x,u}$. Then, according to the definition of the weights matrix, there must exist two profiles $(u, v, m_{-xy}), (u, v, m'_{-xy})$ such that $G(u, v, m_{-xy}) = x$, and $G(u, v, m'_{-xy}) = y$. We define a partition of $K - \{x, y\}$ into three sets: $S = \{t \in K - \{x, y\} : m_t > m'_t\}$, $S' = \{t \in K \setminus \{x, y\} : m_t = m'_t\}$ and $K - S \cup \{h, l\} = \{t \in K - \{x, y\} : m_t < m'_t\}$. We now define the following profile of supports: $(u, v, m'_S, m_{S'}, m_{-(S\cup S'\cup\{x,y\})})$. Because the sets of votes V_i are *regular* for all $i \in N$, we can ensure the existence of such a profile. By applying *candidate monotonicity* from (u, v, m_{-xy}) to $(u, v, m'_S, m_{-(S\cup\{x,y\})})$, we demonstrate that $G(u, v, m'_S, m_{-(S\cup\{x,y\})}) = x$; by applying the same property from (u, v, m'_{-xy}) to $(u, v, m'_S, m_{-(S\cup\{x,y\})})$, we demonstrate that $G(u, v, m'_S, m_{-(S\cup\{x,y\})}) = y$, which is a contradiction.

Finally, we must still prove that the qualified voting scheme Q^ω coincides with the voting scheme g. However, this is immediately obtained from the way in which we have defined the weights matrix ω. □

Regarding weak neutrality, some qualified voting schemes satisfy this property, but others do not. The following result states that, if we require weak neutrality in addition to candidate monotonicity and candidate anonimity, then only the majority voting schemes meet our criteria.

Theorem 5.2. *Let (V^N, g) be a voting mechanism such that V_i is regular for all $i \in N$. Then, g satisfies candidate anonymity, candidate monotonicity, and weak neutrality if and only if g is a majority voting scheme.*

Proof. Any majority voting scheme clearly satisfies the above three properties. Let us show the converse. Let g be a voting scheme that fulfills candidate anonymity, candidate monotonicity, and weak neutrality. We know from Theorem 5.1 that g must be a qualified voting scheme. Suppose, by contradiction, that $g = Q^\omega$ where $\omega \notin \overline{\Omega}$, i.e., that there exist two entries in the weights matrix ω such that $\omega_{x,p} < \omega_{y,r}$ for some $x, y \in K$ and $p, r \in \{1, \ldots, n+1\}$ with $p > r$, and that there exists a voting configuration v_N such that $m_x(v_N) = p$, $m_y(v_N) = r$, and $y = g(v_N)$.

Because V^N is regular, there exists a voting configuration $v'_N \in V^N$ such that $m_x(v'_N) = m_y(v_N)$, $m_y(v'_N) = m_x(v_N)$, and $m_z(v'_N) = m_z(v_N)$ for all $z \in K \smallsetminus \{x, y\}$. Because g is *candidate anonymous* and *weakly neutral*, $g(v'_N) = x$. Now, let $v''_N \in V^N$ be a voting configuration such that $m_y(v''_N) = m_y(v_N)$, $m_x(v''_N) = m_x(v_N)$, and $m_z(v''_N) = 0\ \forall z \in K - \{x, y\}$. We know that such a voting configuration exists because the sets of votes are *regular*. Because g is *candidate monotonic* and *candidate anonymous*, when passing from v_N to v''_N, we now that $g(v''_N) = y$. Conversely, through *candidate monotonicity* and *candidate anonymity*, it should happen that $g(v''_N) = x$ when passing from v'_N to v''_N, which contradicts the fact that $g(v''_N) = y$. □

The following examples show that Theorems 5.1 and 5.2 are tight.

Example 5.1. Let V^N be a profile of sets of votes such that V_i is regular for all $i \in N$. There is a voting scheme that is candidate anonymous and candidate monotonic, but not weakly neutral: Q^ω, where $\omega \in \Omega - \overline{\Omega}$.

Example 5.2. Let V^N be a regular profile of sets of votes such that $V_i = K \cup \phi$ for all $i \in N$. A voting scheme that is weakly neutral and candidate monotonic, but not candidate anonymous is defined as follows. Fix $x \in K$, and let $1 \succ 2 \succ \ldots \succ n$ be an ordering on voters. Then, the *serial dictator voting scheme* associated with x and $\succ$ is defined as follows. For all $v_N \in V^N$

$$g^x_{\succ}(v_N) = \begin{cases} v_j & \text{if } v_j \neq \phi \text{ and } v_i = \phi \text{ for all } i \succ j \\ x & \text{if } v_i = \phi \text{ for all } i \in N \end{cases}$$

Example 5.3. Let V^N be a profile of sets of votes such that $V_i = K \cup \phi$ for all $i \in N$. Let $\overline{\omega} \in \overline{\Omega}$. A voting scheme that is candidate anonymous and weakly neutral but not candidate monotonic is as follows. For all $v_N \in V^N$

$$g^{\overline{\omega}}(v_N) = \arg\min_{x \in K} \overline{\omega}_{x, m_x(v_N)}.$$

Now, we examine to what extent the assumption about the regularity of the sets of votes is a determinant of the two previous results. We present two voting mechanisms in which the sets of votes are not regular and the voting scheme is not a qualified voting scheme (though it is candidate monotonic, candidate anonymous, and weakly neutral).

Example 5.4. Let us assume that $N = \{1, 2, 3\}$ and $K = \{x, y, z, w, h, l, t\}$. Let us consider a voting mechanism (V^N, g) where $V_i = \{\{x, l\}, \{y, w\}, \{y, z\}, \{h, t\}, \{x\}, \{y\}, \{z\}, \{w\}, \{h\}, \{l\}, \{t\}, \phi\}$ for all $i \in N$. Let the voting scheme g be as follows. For all $v_N \in V^N$

$$g(v_N) = \begin{cases} h & \text{if } m_h(v_N) > m_t(v_N) \text{ or } (m_h(v_N) = m_t(v_N) \text{ and } m_w(v_N) \neq 0) \\ t & \text{if } m_h(v_N) < m_t(v_N) \text{ or } (m_h(v_N) = m_t(v_N) \text{ and } m_w(v_N) = 0) \end{cases}$$

The set V_i is not regular because i is not allowed to vote for any combination of two candidates. For instance, she may vote for $\{y, w\}$, but not for $\{x, y\}$, i.e., $V^2 \cap V_i \neq \phi$ but $V^2 \nsubseteq V_i$, which is a condition for regularity. The voting scheme g is candidate monotonic, candidate anonymous, and weakly neutral. Nevertheless, it does not belong to the qualified voting family. Indeed, let $v_N = (\{x, l\}, \{h, t\}, \{y, w\})$ and $v'_N = (\{x, l\}, \{h, t\}, \{y, z\})$. Note that $g(v_N) = h$ and $g(v'_N) = t$ but $M_h(v_N) = \{2\} = M_h(v'_N)$ and $M_t(v_N) = \{2\} = M_t(v'_N)$. Therefore, g cannot be written as a qualified voting scheme.

In reality, there are some electoral processes in which voters may abstain and others in which they are obliged to vote. The election of the President of the United States is of the first type, and the election in Belgium and some South American countries are of the second type. This is closely related to the structure of the sets of votes. By imposing regularity, we are also assuming that $\phi \in V$. This assumption implies that voters, independently of how rich V may be, cannot abstain. The example below considers a profile of sets of votes where the regularity condition is only violated in this respect.

Example 5.5. Let σ be an ordering over the set of candidates K, and let $a = (a_i)_{i=1}^{n-1}$. Let (V^N, g^W) be the voting mechanism in which $V_i = K$ for all $i \in N$ (and therefore $\phi \notin V_i$). The family of generalized median voting schemes can be described using a list of parameters $a = (a_i)_{i=1}^{n-1}$ as $W^a(v_N) = med\{v_1, v_2, ..., v_n; a_1, a_2, .., a_{n-1}\}$. We now show that any W^a satisfies candidate anonymity and candidate monotonicity (but clearly not weak neutrality). It is clear that W^a satisfies candidate anonymity. To show that W^a satisfies candidate monotonicity, it is enough to see that the property only applies when one goes from v_N to v'_N, wherein the chosen candidate, $W^a(v_N)$, strictly increases her support at v'_N while other candidates decrease their support, but in such a case $W^a(v'_N) = W^a(v_N)$. However, whenever $\#K > 3$, the generalized median voting schemes cannot be described as qualified voting schemes.

From Theorems 5.1 and 5.2 and the examples above, we conclude that there is a trade-off between the properties we impose on the voting scheme and the flexibility (measured through regularity) voters have to vote. Our last results show that by strengthening one of the requirements of the voting scheme, we can eliminate the regularity condition. More precisely, if we substitute candidate monotonicity with strong candidate monotonicity, we obtain characterizations that are analogous to Theorems 5.1 and 5.2 without imposing any particular structure on the sets of votes.

Theorem 5.3. *Let (V^N, g) be a voting mechanism. Then, g satisfies candidate anonymity and strong candidate monotonicity if and only if g is a qualified voting scheme.*

Proof. Clearly, any qualified voting scheme satisfies the two desired properties. We argue the converse. Let g be a voting scheme that fulfills candidate anonymity and strong candidate monotonicity. From here, the argument is quite similar to the reasoning of the proof of Theorem 5.1. First, we define a weights matrix ω, and then, we show that $Q^\omega = g$.

Let $G : \mathbb{R}^k \longrightarrow K$ be a mapping that provides the elected candidate taking as input a vector of supports $(m_1, \ldots, m_k)$.

$G(m_1, \ldots, m_k) = x \Leftrightarrow g(v_N) = x$ for some $v_N \in V^N$such that $m_y(v_N) = m_y \ \forall y \in K$

Because g is *candidate anonymous*, G is well-defined. Now, let $\omega \in \mathbb{R}^k \times \mathbb{R}^{n+1}$ be a matrix that solves the following system of inequalities:

(i) $\omega_{x,i} \leq \omega_{x,(i+1)} \ \forall i \in \{1, \ldots, n-1\}$ and $\forall x \in K$.

(ii) $G(m_1, \ldots, m_k) = x$ if and only if $\omega_{x,m_x} > \omega_{y,m_y} \ \forall y \in K - \{x\}$.

We show that the previous conditions do not generate cycles. Assume otherwise, we obtain (directly or indirectly) a situation where $\omega_{x,u} < \omega_{y,v} < \omega_{x,u}$. Then, there exist two profiles $m = (u, v, m_{-xy})$ and $m' = (u, v, m'_{-xy})$ such that $G(m) = x$ and $G(m') = y$. By *strong candidate monotonicity* and *candidate monotonicity*, it must hold that $G(m') \in S(m, m') \cup G(m)$, where $S(m, m') = \{z \in A_g : m_z < m'_z\}$. Inasmuch as $y \notin S(m, m')$, we obtain the desired contradiction. □

Theorem 5.4. *Let (V^N, g) be a voting mechanism. Then, g satisfies candidate anonymity, strong candidate monotonicity and weak neutrality if and only if g is a majority voting scheme.*

Proof. Because it is clear that any majority voting scheme fulfills the above three requirements, we focus on the converse. Let g be a voting scheme that is candidate anonymous, candidate monotonicity, and weakly neutral. We know from Theorem 5.1 that g must be a qualified voting scheme. Suppose, by contradiction, that $g = Q^\omega$ where $\omega \notin \overline{\Omega}$, that is, suppose that there exist two entries in the weights matrix ω such that $\omega_{x,p} < \omega_{y,r}$ for some $x, y \in K$ and $p, r \in \{1, \ldots, n+1\}$ with $p > r$. Then, there exists a voting configuration v_N such that $m_x(v_N) = p$, $m_y(v_N) = r$, and $y = g(v_N)$. □

6 Conclusions

In this paper, we study voting mechanisms, which consist of two elements: a profile of sets of votes (that describes the possible votes that voters can cast) and a voting scheme (that states how to aggregate those votes). We have introduced the qualified voting mechanisms using qualified voting schemes. These voting schemes have an easy and natural functioning. In essence, they select the candidate whose corresponding support has the highest priority, which is determined using a weights matrix. The qualified voting schemes satisfy candidate monotonicity, strong candidate monotonicity, and candidate anonymity. However, Theorem 5.1 has now shown that, when the sets of votes are regular, those are the only voting schemes that are candidate monotonic and candidate anonymous. If, in addition, we impose weak neutrality, then we can only utilize majority voting schemes.

The model in May (1952), where $K = \{x, y\}$ can be adapted to our framework. Each voter i submits her preferences regarding the candidates: x is preferred to y, y is preferred to x, or x is indifferent to y. This is equivalent to the assumption that the sets of votes are $V_i = \{\{x\}, \{y\}, \{x, y\}\}$ for all $i \in N$. The voting scheme selects one winner.[4] The sets V_i are not regular. However, when the set of candidates

[4]Actually, in May (1952) the voting scheme could select both candidates. In such a case we assume that there is a tie-breaking rule.

only consists of two elements, both candidate monotonicity and strong candidate monotonicity coincide. Because of this coincidence, the characterization in May (1952) could be understood as a consequence of Theorem 5.4.

There are two interesting extensions that, although beyond the goals of this paper, are worth mentioning. We have focused on voting situations in which we need to choose a single winner. Although this situation is quite common, there are other scenarios where more candidates have to be selected (e.g., elections in multi-member districts). The properties we consider are easy to extend to this new framework, but the characterization we obtain cannot be directly applied to it. We have to decide whether the goal is to elect a fixed number of candidates (no more and no less) or a maximum number (e.g., three candidates at most). Depending on which goal we set, the nature of the results we obtain may vary significantly.

Another promising extension is adding a third element to the problem: preferences. So far, the voting situation is described by the sets of votes (or messages) and the voting scheme (or outcome function). By introducing preferences, voters are able to compare the results of the election, which opens the door to studying the manipulability of the qualified voting schemes or their potential extensions to a multi-winner framework.

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