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Strong composition down in the adjudication of conflicting claims

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WP 2014/03

STRONG COMPOSITION DOWN IN IN THE ADJUDICATION OF CONFLICTING CLAIMS*

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October 28, 2014

Abstract

This paper is devoted to the study of claims problems. I characterize the family of rules that satisfy both *strong composition down* (robustness with respect to reevaluations of the estate) and *consistency* (robustness with respect to changes in the set of agents). These are the *fixed path rules*, a generalization of the *weighted constrained equal awards rules*. In addition, if *strong composition down* and *consistency* are combined with *homogeneity*, only the *weighted constrained equal awards* rules survive. I also prove that the *constrained equal awards rule* is the only rule that satisfies *strong composition down*, *consistency* and *equal treatment of equals*.

Keywords: claims problem, strong composition down, fixed path rules, constrained equal awards rule.

JEL Classification: D63, D70

*I am very grateful to Carmen Herrero, Antonio Villar, and William Thomson for their helpful and stimulating comments. All errors are my sole responsibility. I acknowledge financial support from the Ministerio de Educación y Ciencia under projects SEJ2007-67734 and SEJ2007-67734, from the Programa de Becas Postdoctorales MEC/Fulbright, Junta de Andalucí under project SEJ1252, from the Generalitat Valenciana under project GV06/275, and finally from the Instituto Valenciano de Investigaciones Económicas.

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1 Introduction

Adjudicating conflicting demands is a very old problem, first modeled by O'Neill (1982). Specifically, a *claims problem* refers to any situation where a limited commodity has to be distributed among agents, but the available amount falls short of the total demand. A canonical illustration is alloting the liquidation value of a bankrupt firm among its creditors. The reader is referred to Thomson (2003) for a broad exposition of the literature.

A claims problem is determined by three elements: a set of *agents*; the amount of resource to be distributed, also called the *estate*; and a vector of agents' demands or *claims*. A *rule* is a way of distributing the estate among the agents according to their claims. In this work, we follow the axiomatic approach, justifying the rules in terms of the properties they satisfy. These properties usually refer to notions of equity and stability.

Among the various properties normally considered when designing rules, *composition down* (Moulin (2000)) emerges as a useful requirement. Imagine that we were too optimistic when estimating the value of the estate; its actual value is smaller than expected. There are now two alternatives. One is to solve a problem based on the original claims and the smaller estate. The other alternative is to adjust each agent's claim to their allocation under the overestimated estate. Composition down requires the final allocation to be independent of this choice. To put it another way, this property assumes that either all agents unanimously demand the original claims, or all agents unanimously demand the awards for the overestimated estate.

This paper revises the composition down assumption, such that some agents demand their original claims while others demand their adjusted claims. The application of the Treaty of Kioto to the European Union -EU- constitutes a good example of the alternative formulation we propose. Originally, the European Commission -EC- distributed the CO_2 emissions -the estate- among the EU members. Due to the pressure of the public opinion and others international agreements, a new reduction in the aggregate European emission was imposed, and the EC has to recalculate the new quotas. Suppose that some countries demand the original claims, while some others demand the tentative assigned quota -maybe because they already started to apply the reduction in their respective states-. *Strong composition down* states that no matter who (which agents) demands what (the original claim or the tentative allocation), the distribution of the reduced estate is the same.

We also consider a standard property that provides stability with respect to changes in the set of agents. *Consistency*, first introduced by Aumann and Maschler (1985), refers to situations where a tentative distribution of the estate has been made, and one or more agents leave after accepting their awards. Consistency states that the reduced problem should be solved in such a way that all remaining agents receive the same amount that they did in the tentative allocation.

One of the most widely studied rules is the so-called *constrained equal awards rule* (Maimonides, 12th Century). It proposes that all individuals should be treated equally, so long as no one receives more than their claim. For any claims vector, the agents who do not receive their full claim receive equal amounts, no matter how small or large the estate. A generalization of this idea underlies the *weighted constrained equal awards rules*, which favor agents who are perceived as more deserving. These rules are parametrized by a vector of positive weights. The agents who do not receive their full claim receive amounts that are equal when divided by their respective weights, no matter how small or large the estate. In this paper I consider a third class of solutions: *the fixed path rules*.[1] Rules in the fixed-path family respect the spirit of the first two rules mentioned, but the amounts received by agents who do not get their full claim do depend on the size of the estate.

Our main result is that the only rules satisfying both strong composition down and consistency are fixed path rules (Theorem 3.1). Moulin (2000) characterized rules fulfilling homogeneity, consistency, composition down, and *composition up* (the dual of composition down, where the estate is underestimated).[2] This family of rules, called the $\mathcal{M}$ family, is extremely large; its solutions run the gamut from proportional to dictatorial. Interestingly, if we merely strengthen composition down to strong composition down in Moulin's assumptions, we reduce the $\mathcal{M}$ family to the family of weighted constrained equal awards rules (Theorem 4.1).

Finally, keeping in mind the fact that any claims problem describes a rationing situation, we need an impartiality requirement. *Equal treatment of equals* is a very mild constraint, which simply states that agents with identical claims receive equal awards. Young (1987) showed that the only continuous rules satisfying both consistency and equal treatment of equals are the so-called *parametric rules*. I find that there are also solutions within the parametric family fulfilling strong composition down. Corollary 4.1 shows that there is actually only one solution: the constrained equal awards rule.

The rest of the paper is structured as follows: In Section 2 I set up the model and present the fixed path family. In Section 3 I define strong composition down and derive my main result. In Section 4 I explore other properties fulfilled by the fixed path rules, and provide alternative characterizations of the weighted and unweighted constrained equal awards rules. Section 5 concludes with some comments and remarks. The proofs are relegated to an appendix.

2 Statement of the model. The fixed path rules.

Let $\mathbb{N}$ be the set of all potential agents. Let $\mathcal{N}$ denote the family of all finite subsets of $\mathbb{N}$. Let $\gamma \in \mathbb{R}$ be an upper bound on the agents' demand.[3] In a claims problem, hereafter simply a **problem**, a given estate $E \in \mathbb{R}_{++}$ has to be distributed among a group of agents $N \in \mathcal{N}$ subject to their claims. The vector of claims is denoted $c = (c_i)_{i \in N} \in [0, \gamma]^N$. Furthermore, E is not large enough to fully satisfy all the claims. Therefore, a problem is

[1] These rules were first introduced by Moulin (1999) in the context of resource allocation with single-peaked preferences. I adapt them to claims problems, but keep the name.

[2] Thomson (2006a) also provides a simple and illustrative proof of this result.

[3] This upper bound is not usual in the literature on claims problems. Adding it simplifies the presentation and makes for much more illustrative proofs. However, all the results and considerations discussed in this paper are unchanged if the assumption is removed.

defined by the triple $e = (N, E, c)$, where $\sum_{i \in N} c_i \geq E$ and $c_i \leq \gamma$ for each $i \in N$.

We denote by $\mathbb{C}^N$ the class of claims problems with a fixed population N, and by $\mathbb{C}$ the class of all claims problems:

$$\mathbb{C}^N = \left\{ e = (N, E, c) \in \{N\} \times \mathbb{R}_{++} \times [0, \gamma]^N : \sum_{i \in N} c_i \geq E \right\}$$

and

$$\mathbb{C} = \bigcup_{N \in \mathcal{N}} \mathbb{C}^N.$$

An **awards vector** for $e \in \mathbb{C}$ is a division of the estate among the agents, that is, it is a list $\boldsymbol{x} \in \mathbb{R}^N_+$ such that: (a) Each agent receives a non-negative amount which is not larger than her claim (for each $i \in N$, $0 \leq x_i \leq c_i$); and (b) the estate is exactly distributed ($\sum_{i \in N} x_i = E$). Let $\boldsymbol{X(e)}$ be the set of all awards vectors for $e \in \mathbb{C}$. A **rule** is a way of selecting awards vectors, that is, it is a continuous function, $\boldsymbol{S} : \mathbb{C} \longrightarrow \bigcup_{e \in \mathbb{C}} X(e)$, that selects, for each problem $e \in \mathbb{C}$, a unique awards vector $S(e) \in X(e)$.

Let S be a rule and let c be a fixed claims vector. I denote by $p_S(c)$ the sequence of awards vectors produced by $S(N, E, c)$ as the estate E varies from 0 to C. This sequence is called the **path of awards** of $\boldsymbol{S}$ for $\boldsymbol{c}$. A rule can be defined by specifying the paths of awards of possible claims vectors.

Below I define two of the most prominent rules in the literature. Each one subscribes to a different idea of fairness in the distribution of the estate. The first follows an Aristotelian notion of justice, and proposes a distribution of the estate proportional to the claims.

Proportional rule, ***prop***: For each $e \in \mathbb{C}$, it selects the unique awards vector $prop_i(e) = \lambda \cdot c_i$ for some $\lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} \lambda \cdot c_i = E$.

The second rule comes from Maimonides (12th century). It states that agents should be treated equally, independently of differences in their claims. Thus, the so-called *constrained equal awards rule* proposes equality in gains, adjusting, if necessary, to ensure that no agent receives more than her claim.

Constrained equal awards rule, ***cea***: For each $e \in \mathbb{C}$, it selects the unique awards vector $cea_i(e) = \min\{c_i, \lambda\}$ for some $\lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} \min\{c_i, \lambda\} = E$.

Now consider the family of so-called *weighted constrained equal awards rules*. As its name suggests, this family is a generalization of the *cea* rule. Under *cea*, the agents are fully equivalent. But it may happen that differences between the agents call for some adjustments. These differences are represented by mean of a vector of weights. For each $i \in \mathbb{N}$, let $\alpha_i \in \mathbb{R}_{++}$ be agent i's weight and denote by $\alpha = (\alpha_i)_{i \in \mathbb{N}}$ the vector of weights.

Constrained equal awards rule with weights $\boldsymbol{\alpha = (\alpha_i)_{i \in \mathbb{N}}}$, $\mathbf{cea}^{\boldsymbol{\alpha}}$: For each $e \in \mathbb{C}$, it selects the unique awards vector $cea^{\alpha}_i(e) = \min\{c_i, \alpha_i \lambda\}$ for some $\lambda \in \mathbb{R}_+$ such that $\sum_{i \in N} \min\{c_i, \alpha_i \cdot \lambda\} = E$.

Figure 1 illustrates the three rules by showing paths of awards for several claims vectors.

The symbol Γ denotes the claims vector whose components are all equal to γ: $\Gamma = (\gamma, \ldots, \gamma)$.

[Insert Figure 1 about here.]

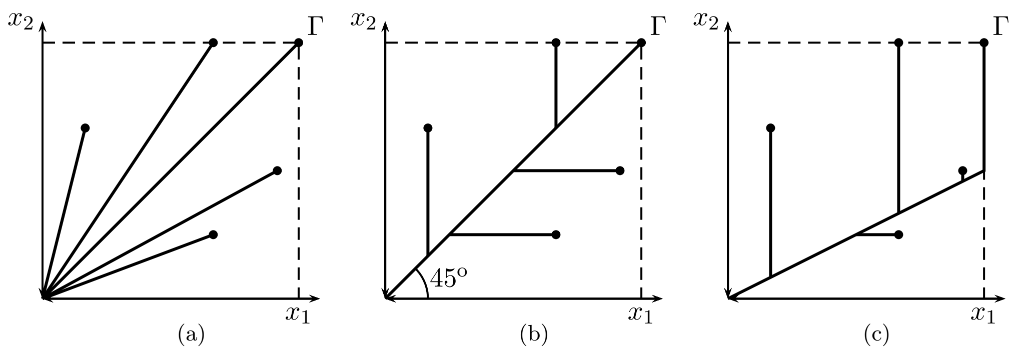

Figure 1: Path of awards for different claims vectors in two-agent problems. (a) Proportional rule. (b) Constrained equal awards rule. (c) Weighted constrained equal awards rule for $\alpha = (2, 1)$.

Now I will present the *fixed path rules*. This class of solutions was first introduced by Moulin (1999) in the context of resource allocation with single-peaked preferences. I adapt the concept to claims problems. For now, let us focus on a two-agent framework: $N = \{i, j\}$. The typical path of awards of a fixed path rule can be described as follows. There is a curve leading from the origin to Γ (the *fixed path*), and there are vertical and horizontal rays extending from the fixed path to the claims vectors. As Figure 1 shows, the *cea* and $cea^{(2,1)}$ rules are both fixed path rules. The proportional rule is not.

Before providing a formal description of the fixed path rules, I introduce some auxiliary notions. Given a specific set N of n agents, an ***N*-path** is a mapping $\boldsymbol{p^N} : [0, n\gamma] \longrightarrow \mathbb{R}^N$ such that (a) p^N is monotonic (for each $z, z' \in [0, n\gamma]$, if $z \leq z'$ then $p^N(z) \leq p^N(z')$); and (b) for each $z \in [0, n\gamma]$, $\sum_{i \in N} p_i^N(z) = z$. Let $\boldsymbol{\pi(p^N)}$ be the curve on $\mathbb{R}^N$ that p^N draws as z varies from 0 to $n\gamma$. A *fixed path* rule specifies an N-path for each set $N \in \mathcal{N}$, with the requirement that all N-paths be *projectionally consistent*.

A fixed path, $\boldsymbol{p}$, is a collection of mappings $\{p^N\}_{N \in \mathcal{N}}$ such that if $N \subseteq N'$, then $\pi(p^{N'})_N = \pi(p^N)$.[4]

A different *fixed path rule* is associated with each fixed path. The collection of such rules is the *fixed path family*.

Fixed path rule for $\boldsymbol{p}$, $\boldsymbol{F^p}$: For each $e = (N, E, c) \in \mathbb{C}$, it selects the unique awards

[4]Notice that $\pi(p^{N'})$ and $\pi(p^N)$ are two curves on two different spaces, $\mathbb{R}^{N'}$ and $\mathbb{R}^N$. The notation $\pi(p^{N'})_N$ means the projection of $\pi(p^{N'})$ onto the space $\mathbb{R}^N$.

vector $F^p(e) = \min\{c, \lambda^p\}$ for some $\lambda^p \in p^N$ such that $\sum_{i \in N} \min\{c_i, \lambda_i^p\} = E$.[5]

Figure 2 illustrates the fixed path family with three examples.

[Insert Figure 2 about here.]

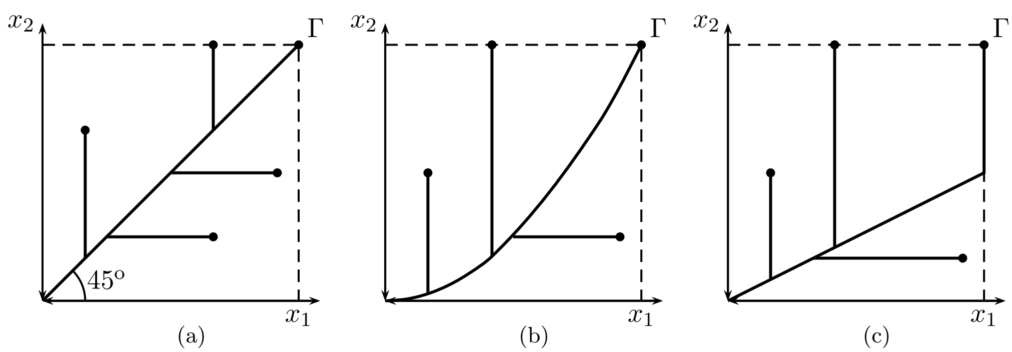

Figure 2: Paths of awards of three rules in the fixed path family. Case (a) is the constrained equal awards rule, where $p^N(z) = \left(\frac{z}{2}, \frac{z}{2}\right)$. Case (b) is the F^p rule in Example 2.1. Case (c) is a weighted constrained equal awards rule, where $p^N(z) = \left(\frac{2}{3}z, \frac{1}{3}z\right)$ if $z \leq \frac{3\gamma}{2}$, and $p^N(z) = (\gamma, z - \gamma)$ otherwise. All three rules have a fixed path from $(0, 0)$ to $\Gamma = (\gamma, \gamma)$.

Example 2.1. Let p be a fixed path for the two-agent case, $p^{\{i,j\}}(z) = \left(\frac{z}{2}, \frac{z^2}{4\gamma}\right)$ with $z \in [0, 2\gamma]$. (This is Case b in Figure 2.) The following table shows how the *cea*, $cea^{(2,1)}$, and F^p rules resolve various claims problems.

		Rules ($\gamma = 100$)		
c	E	cea	$cea^{(2,1)}$	F^p
$(30, 9)$	6	$(3, 3)$	$(4, 2)$	$(5.67, 0.32)$
$(30, 9)$	24	$(15, 9)$	$(16, 8)$	$(20, 4)$
$(30, 9)$	28	$(19, 9)$	$(19, 9)$	$(22.80, 5.19)$
$(15, 40)$	9	$(4.5, 4.5)$	$(6, 3)$	$(8.30, 0.69)$
$(15, 40)$	24	$(12, 12)$	$(16, 8)$	$(15, 9)$

3 Two properties: strong composition down and consistency.

The next property has been widely studied in the literature on claims problems, and is particularly useful when some uncertainty over the estate exists. Let $e = (N, E', c)$ be

[5]It is worth noting that for the weighted and non-weighted constrained equal awards rules, λ was a scalar. In this definition, λ^p is an n-dimensional point on the N-path p^N.

the problem to solve, and let $x = S(e)$ be the awards vector selected by a rule S for that problem. Imagine that we were too optimistic in estimating the value of the estate, so its actual value E is smaller than expected: $E < E'$. Two claims vectors now arise as legitimate demands. The first is the original claims vector c, and the second is the vector of promised awards x. *Composition down* requires that we end up with the same distribution of E whether c or x is used as the claim vector.[6]

Composition down: For each $e \in \mathbb{C}$ and each $E' \in \mathbb{R}_+$ such that $C > E' > E$, $S(e) = S(N, E, S(N, E', c))$.

Now, composition down considers only the choice between c and x. When imposing this property, one tacitly assumes that c and x are the only *legitimate* claims; either the agents unanimously demand their original claims c, or they unanimously demand their promised awards x. I now define a property that recognizes intermediate situations, where no consensus is required.

Let $T \subseteq N$ be a subset of the agents in N. Let $(x_T, c_{N \setminus T})$ be the claims vector where agents in T ask for their promised awards x and agents not in T ask for their original claims c. Remaining within the spirit of composition down, *strong composition down* requires that we end up with the same allocation regardless of which claims vector we consider. That is, the legitimate choices are now x, c, and $(x_T, c_{N \setminus T})$ for any $T \subseteq N$.

Strong composition down: For each $e \in \mathbb{C}$, each $E' \in \mathbb{R}_+$ such that $C > E' > E$, and each $T \subseteq N$, $S(e) = S(N, E, (S_T(N, E', c), c_{N \setminus T}))$.

Strong composition down is actually very demanding, even though there are many rules that satisfy this property (in particular, the fixed path family).

As I mentioned in the Introduction, the Treaty of Kioto and subsequent protocols are illustrative examples of a possible application of strong composition down. It is well-known that the European Commission has to divide the European quota of the CO_2 emissions among the EU countries. Let us assume that the claims c_i are the issued tons in 2000 by country i. The overall quota for the whole EU is E' (which is obviously smaller than $\sum_{i \in N} c_i$ because this is the goal of the international agreements). Let $x = S(N, E', c)$ be the emissions vector. Months after, new international commitments force to reduce the overall European quota to $E < E'$. Since each country was designing its particular national policy conditioned to allocation x, it is quite reasonable to allow each of them to choose whether to demand c_i or x_i.

Now let us consider a procedural property related to changes in the agent set. Suppose that after solving a problem, some agents leave with their awards. The remaining agents then re-evaluate how to allocate the smaller estate among themselves. *Consistency* requires that each of these agents should receive the same amount that they were allocated before the re-evaluation.[7]

Consistency: For each $e = (N, E, c) \in \mathbb{C}$, each $N' \subset N$, and each $i \in N'$, $S_i(e) =$

[6]This property was formulated by Moulin (2000).

[7]This property has been widely studied; see Thomson (1998) for a survey.

$S_i(N', \sum_{j \in N'} S_j(e), c_{N'})$.

Consider a rule S that coincides with constrained equal awards when $N = \{1, 2, 3\}$, and coincides with a dictatorial rule when $N = \{1, 2\}$. In the latter case, the rule favors agent 1. Comparing agent 1 to agent 2, we find that this rule is quite fair for agent 2 when agent 3 is present, but extremely unfair to agent 2 otherwise. Consistency avoids this type of drawback. All of the rules defined in the previous section are consistent.

Now I state my main result, a theorem identifying the family of rules that satisfy both strong composition down and consistency. Its proof is given in the appendix.

Theorem 3.1. *The fixed path rules are the only rules satisfying both strong composition down and consistency.*

4 Some implications

This section explores some other axioms, and discusses the effect of combining each one with strong composition down and consistency.

Homogeneity stipulates that problems in which the claims and estate are small should be treated similarly to problems where the claims and estate are large. Although this criterion cannot always be applied, it is considered very desirable in many situations. Formally, homogeneity means that if both the estate and the claims vector are multiplied by the same positive amount, the awards vector will be multiplied by the same amount.

Homogeneity. For each $e = (N, E, c) \in \mathbb{C}$ and each $\beta \in \mathbb{R}_+$, $S(N, \beta E, \beta c) = \beta S(N, E, c)$.

All weighted constrained equal awards rules are homogenous. Some of the fixed path rules satisfy homogeneity, but others do not.

The *composition up* property is similar to composition down and equally useful. Let $e = (N, E', c)$ be a problem to solve, and let $x = S(e)$ be the solution of rule S for that problem. Imagine that we were too pessimistic when estimating the value of the estate, so the actual value E is greater than expected $(E > E')$. Two possible courses of action are evident. One is to solve the problem with the revised estate, $S(N, E, c)$. The other is to first assign the original awards x, then allocate the remaining estate $(E - E')$ after reducing the claims by the amounts just given $(c \to c - x)$. Composition up states that both approaches should result in the same awards.[8]

Composition up. For each $e \in \mathbb{C}$ and each $E' \in \mathbb{R}_+$ such that $E' < E$, then $S(e) = x + S(N, E - E', c - x)$, where $x = S(N, E', c)$.

All weighted constrained equal awards rules satisfy composition up. Some of the fixed path rules satisfy this property, but others do not.

Moulin (2000) characterized the family of rules that satisfy composition down, composition up, homogeneity, and consistency. This family is called $\mathcal{M}$, and it contains a wide variety

[8]This property was formulated by Young (1988).

of rules. The proportional rule and weighted constrained equal awards both belong to $\mathcal{M}$, as do the weighted constrained equal losses rule, dictatorial rules, and combinations thereof.[9]

Which are the implications of substituting strong composition down for composition down in the characterization result in Moulin (2000)? The answer is that only cea^{α} rules survive (Theorem 4.1 below). That is, these are the only rules that satisfy strong composition down, composition up, homogeneity, and consistency. By introducing strong composition down, we separate the weighted constrained equal awards rules from the broad family characterized in Moulin (2000). The proof of Theorem 4.2 is given in the appendix.

Theorem 4.1. *The weighted constrained equal awards rules are the only rules satisfying*

(a) Strong composition down, composition up, and consistency.
(b) Strong composition down, homogeneity, and consistency.

Furthermore, as Theorem 4.1 suggests, (a) homogeneity can be obtained as a consequence of strong composition down, composition up, and consistency; and (b) composition up can be obtained as a consequence of strong composition down, homogeneity, and consistency.

Now I introduce a minimal requirement of fairness. *Equal treatment of equals* is a very mild constraint, simply requiring that agents with identical claims receive equal rewards.

Equal treatment of equals. For each $e \in \mathbb{C}$ and each $\{i, j\} \subseteq N$, if $c_i = c_j$ then $S_i(e) = S_j(e)$.

By adding this property to the axioms of Theorem 3.1, we obtain a new characterization of the constrained equal awards rule.

Corollary 4.1. *The constrained equal awards rule is the only rule satisfying equal treatment of equals, strong composition down, and consistency.*

Young (1987) introduced the family of *parametric rules*, which can be defined as follows. Let $f : \mathbb{R}_+ \times [a, b] \longrightarrow \mathbb{R}_+$ be a continuous function such that (a) f is non-decreasing in the second argument, and (b) for each $x \in \mathbb{R}_+$, $f(x, a) = 0$ and $f(x, b) = b$. Let Φ denote the class of such functions. For each $f \in \Phi$, we can define a parametric rule.

Parametric rule for $\boldsymbol{f}$, $\boldsymbol{S^f}$. For each $e \in \mathbb{C}$, it selects the unique awards vector $S_i^f(e) = f(c_i, \lambda)$ for some $\lambda \in [a, b]$ such that $\sum_{i \in N} f(c_i, \lambda) = E$.

Young (1987) proved that the *parametric rules* are the only rules fulfilling both equal treatment of equals and consistency. Corollary 4.1 implies that there is only one parametric rule that also satisfies strong composition down. Therefore, by adding strong composition down to Young (1987), we separate the constrained equal awards rule from the parametric family.

[9] In order to avoid the necessary technicalities I do not introduce here the formal definition of the family (see Moulin (2000)). The name comes from Thomson (2003), who provides a very illustrative definition of the $\mathcal{M}$ family.

5 Final remarks

This work introduced the property of strong composition down, which generalizes composition down to mixed claims vectors. I also characterized the family of fixed path rules for claims problems as a generalization of the weighted constrained equal awards family. Taken together, strong composition down and consistency define the fixed path family. By adding either homogeneity or composition up, we reduce the rules space to the weighted constrained equal awards family. After adding equal treatment of equals, only the constrained equal awards rule survives. Interestingly, strong composition down is enough to separate the cea^{α} and cea rules in Moulin (2000) and Young (1987) respectively.

This paper implicitly considered rules from the point of view of gains. Nevertheless, the dual analysis is also quite common in the literature. Two rules are dual to each other if one assigns awards in the same way that the other assigns losses. The duals of the constrained equal awards and weighted constrained equal awards rules are the *constrained equal losses* and *weighted constrained equal losses rules* respectively.[10] Similarly, we may define the dual of the fixed path family.

The notion of duality also applies to properties. Two properties are dual if when a rule satisfies one property, the dual of the rule satisfies the other. Homogeneity and equal treatment of equals are self-dual (the dual property is itself), while the dual of composition down is composition up. We may consider the dual property of strong composition down -*strong composition up*-, which relates composition up in the same way that strong composition down relates to composition down. In light of Theorem 3.1, and using the characterization-by-duality result of Herrero and Villar (2001), the dual of strong composition down and consistency characterize the dual of the fixed path family.

[10]Refer to Thomson (2003) for a formal description of both.

Appendix A. Proofs

This appendix provides the proofs of Theorems 3.1 and 4.1, preceded by some definitions and technical results.

Resource monotonicity stipulates that no agent should be penalized as a consequence of an increase in the estate.

Resource monotonicity. For each $(N, E, c) \in \mathbb{C}$, if $E' > E$ then $S(N, E', c) \geq S(N, E, c)$.

Let us consider a problem and one of its awards vectors with the following feature. For each two-agent subset of the agents involved, the rule chooses the restriction of the original awards vector to the two-agent subset. *Converse consistency* requires that the allocation be the one selected by the rule for the original problem.[11]

Let $c.con(e; S) \equiv \{x \in \mathbb{R}^N_+ : \sum_{i \in N} x_i = E$ and for all $N' \subset N$ such that $|N'| = 2, x_{N'} = S(N', \sum_{i \in N'} x_i, c_{N'})\}$

Converse consistency. For each $e \in \mathbb{C}$, $c.con(e; S) \neq \phi$, and if $x \in c.con(e; S)$, then $x = S(e)$.

Lemma 5.1 (Elevator Lemma, Thomson (1998)). *If a rule S is consistent and coincides with a conversely consistent rule S' in the two-agent case, then it coincides with S' in general.*

Proposition 5.1 (Chun (1999)). *Resource monotonicity and consistency together imply converse consistency.*

Proof of Theorem 3.1.

It is not difficult to check that any fixed path rule satisfies *strong composition down* and *consistency*. It is also straightforward that *strong composition down* implies *composition down* and that the latter implies *resource monotonicity*. Therefore, by Proposition 5.1, any fixed path rule is *converse consistent*. Let S be a rule fulfilling *strong composition down* and *consistency*. We define the N-path as the path of awards of S for claims vector Γ_N as the estate varies from 0 to $n\gamma$. Let p be the fixed path once we consider the N-paths for all $N \in \mathcal{N}$, which is well defined because S is *consistent*. We show that $S = F^p$. By the Elevator Lemma, it is enough to prove the result in the two-agent case. Let $N = \{i, j\}$ and $c \in \mathbb{R}^{\{i,j\}}$ such that $c_i \leq c_j$. Let $y \in p(\Gamma) = p^N$ be a vector such that $y_i = c_i$ and $y_j < c_j$. Let $z \in p(\Gamma)$ be a vector such that $z_j = c_j$ and $z_i > c_i$. I now distinguish several cases.

Case 1: If $c \in p(\Gamma)$. Let $E \in [0, c_i + c_j]$. Note that since $c \in p(\Gamma)$, $S(N, c_i + c_j, \Gamma) = c$ by the definition of an path of awards. Then $S(N, E, c) = S(N, E, S(N, c_i + c_j, \Gamma))$. By *strong composition down*, $S(N, E, S(N, c_i + c_j, \Gamma)) = S(N, E, \Gamma) = F^p(N, E, \Gamma) = F^p(N, E, c)$. Therefore, $S(N, E, c) = F^p(N, E, c)$.

[11] This property was formulated by Chun (1999).

Case 2: If $c \notin p(\Gamma)$ and $E \leq y_i + y_j$. By Case 1, $S(N, y_i + y_j, z) = F^p(N, y_i + y_j, z) = y$. Then, notice that $c = (y_i, z_j) = (S_i(N, y_i + y_j, z), z_j)$. By *strong composition down*, $S(N, E, z) = S(N, E, (S_i(N, y_i + y_j, z), z_j)) = S(N, E, c)$. Since $z \in p(\Gamma)$, by Case 1, we know that $S(N, E, z) = F^p(N, E, z) = F^p(N, E, c)$. Therefore, $S(N, E, c) = F^p(N, E, c)$.

Case 3: If $c \notin p(\Gamma)$ and $E > y_i + y_j$. On the one hand, by *strong composition down*, $S(N, y_i + y_j, c) = S(N, y_i + y_j, S(N, E, c)) \leq S(N, E, c)$; on the other hand, $S(N, y_i + y_j, c) = F^p(N, y_i + y_j, c) = y$. Thus, $c \geq S(N, E, c) \geq y$. Therefore, $S(N, E, c) = F^p(N, E, c)$.

Therefore, S and F^p coincide in the two-agent case, and thus they do so in general.

[Insert Figure 3 about here.]

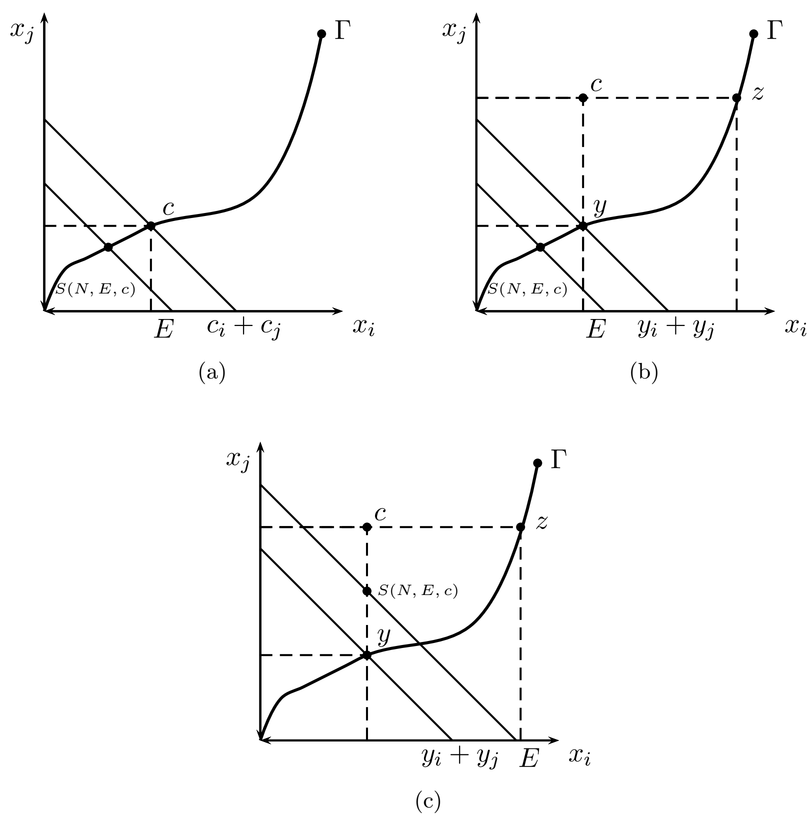

Figure 3: Illustration of the proof for the two-agent case. (a) Case 1. (b) Case 2. (c) Case 3.

Remark 5.1. The proportional rule (see Section 2 for its definition) satisfies consistency and composition down but violates strong composition down. A rule satisfying strong composition down but not consistency can be defined as follows.

$$S(e) = \begin{cases} cea^{(2,1)}(e) & \text{if } N = \{1, 2\} \\ cea(e) & \text{otherwise} \end{cases}$$

The properties that characterize the fixed path family are therefore independent.

Proof of Theorem 4.1.

(a) The weighted constrained equal awards rules trivially fulfill the three properties. We only need to prove the converse for the two-agent case, $N = \{i, j\}$, since the Elevator Lemma extends this result to the general case. Let S be a rule satisfying *strong composition down*, *composition up*, and *consistency*. By Theorem 3.1, S is a fixed path rule, and S is completely described by its paths. Let $\alpha \in \mathbb{R}^2$ be the point where the path $p(\Gamma)$ reaches the square defined by the four vertices $(0,0)$, $(0,\gamma)$, $(\gamma, 0)$ and $(\gamma,\gamma) = \Gamma$ (see Figure 4). Without loss of generality, we can assume that $\alpha_j = \gamma$. Below I prove that if $z \in p(\Gamma)$ then $\alpha - z \in p(\Gamma)$, which implies that the path of awards from 0 to α is a straight line. For any $z, z' \in \mathbb{R}^2$, let $\overline{zz'}$ denote the segment joining z and z'. I distinguish several cases.

Case 1: $p(\Gamma)$ is always above the segment $\overline{0\alpha}$ (see Figure 4(a)). Let $z \in p(\Gamma)$, so $S(N, z_i + z_j, \alpha) = z$. Since S is a fixed path rule and z is above $\overline{0\alpha}$ There exists $E \in \mathbb{R}_{++}$ such that $S(N, E, \alpha - z) = (E - \alpha_j + z_j, \alpha_j - z_j)$ and $E - \alpha_j + z_j < \alpha_i - z_i$. By *composition up,*

$$\begin{aligned} S(N, E + (z_i + z_j), \alpha) &= S(N, z_i + z_j, \alpha) + S(N, E, \alpha - z) \\ &= (z_i, z_j) + (E - \alpha_j + z_j, \alpha_j - z_j) \\ &= (E + z_i + z_j - \alpha_j, \alpha_j) \\ &= (E + z_i + z_j - \gamma, \gamma) \end{aligned}$$

Since we imposed $E - \gamma + z_j < \alpha_i - z_i$, we obtain $E + z_i + z_j - \gamma < \alpha_i$. This contradicts the definition of α.

Case 2: $p(\Gamma)$ is sometimes above and sometimes below the segment $\overline{0\alpha}$ (see Figure 4(b)). Let $z \in p(\Gamma)$ be the point where, for instance, $p(\Gamma)$ first crosses the segment $\overline{0\alpha}$ from above to below (a similar argument can be applied for another crossing point). By *composition up*, for $\varepsilon > 0$,

$$\begin{aligned} S(N, \varepsilon + z_i + z_j, \alpha) &= S(N, z_i + z_j, \alpha) + S(N, \varepsilon, \alpha - z) \\ &= z + S(N, \varepsilon, \alpha - z) \end{aligned}$$

Note that $S(N, \varepsilon, \alpha - z)$ is above $\overline{0\alpha}$, and so is $S(N, \varepsilon + z_i + z_j, \alpha)$. This contradicts the definition of z.

Case 3: $p(\Gamma)$ is always below the segment $\overline{0\alpha}$, and there exists a point β as in Figure 4(c). Using *composition up*, we find that for each $E \in [\beta_i + \beta_j, 2\gamma]$,

$$\begin{aligned} S(N, E, \Gamma) &= S(N, \beta_i + \beta_j, \Gamma) + S(N, E - (\beta_i + \beta_j), \Gamma - \beta) \\ &= \beta + S(N, E - (\beta_i + \beta_j), \Gamma - \beta) \end{aligned}$$

Then $S(N, E - (\beta_i + \beta_j), \Gamma - \beta) = S(N, E, \Gamma) - \beta$ for all $E \in [\beta_i + \beta_j, 2\gamma]$. That means that the path of awards of S for $\Gamma - \beta$ is as shown in Figure 4(c). But this contradicts the definition of a fixed path rule.

Case 4. Finally, we will show that point β of Case 3 always exists (see Figure 4(d)). Suppose that it does not. Let $z \in p(\Gamma)$, $z = S(N, z_i + z_j, \alpha)$. For $E \in (z_i + z_j, \alpha_i + \alpha_j)$, by *composition up*, we have

$$\begin{aligned} S(N,E,\alpha) &= S(N, z_i + z_j, \alpha) + S(N, E - (z_i + z_j), \alpha - z) \\ &= z + S(N, E - (z_i + z_j), \alpha - z) \end{aligned}$$

It is not difficult to check that $\alpha - z$ is above the segment $\overline{0\alpha}$. Since S is a fixed path rule and $\alpha - z$ is above $\overline{0\alpha}$, there exits $E' \in (z_i + z_j, \alpha_i + \alpha_j)$ such that

$$S(N, E' - (z_i + z_j), \alpha - z) = (\alpha_i - z_i, E' - (z_i + z_j) - (\alpha_i + \alpha_j)).$$

Therefore,

$$\begin{aligned} S(N,E',\alpha) &= z + S(N, E - (z_i + z_j), \alpha - z) \\ &= (z_i, z_j) + (\alpha_i - z_i, E' - (z_i + z_j) - (\alpha_i - z_i)) \\ &= (\alpha_i, E' - \alpha_i) \end{aligned}$$

But we can now define β as $\beta = (\alpha_i, E' - \alpha_i)$.

[Insert Figure 4 about here.]

(b) This proof comes from Thomson (2006b), who characterized the set of homogeneous rules. Using their result, it is not difficult to check that the only homogeneous rules within the fixed path family are the weighted constrained equal awards rules. Alternatively, one may use an argument similar to Case (a) to show that the fixed path should be like the weighted constrained equal awards rules.

Proof of Corollary 4.1.

By Theorem 3.1, it is enough to show that the constrained equal awards rule is the unique fixed path rule satisfying *equal treatment of equals*. A rule satisfies *equal treatment of equals* if and only if the diagonal is its path of awards vector Γ; that is, if $(\lambda, \ldots, \lambda) \in p^N$ for all $\lambda \leq \gamma$. Therefore, $F^p = cea$.

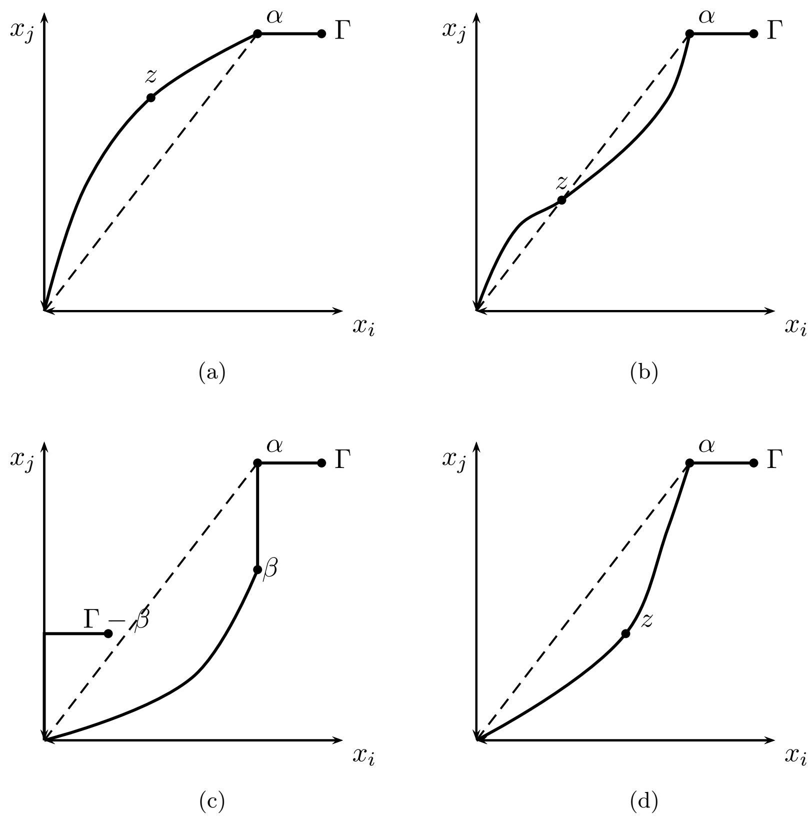

Figure 4: Illustration of the proof for the two-agent case. (a) Case 1. (b) Case 2. (c) Case 3. (d) Case 4.

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