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FEEDBACK ACTIONS ON LINEAR PARAMETER-VARYING SYSTEMS

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Abstract
Linear parameter-varying systems are studied by means of geometrical translation of some recent results of algebraic nature dealing with the feedback actions on linear control systems.

Key words
Feedback invariant, vector bundle.

1 Introduction

Parametric-varying linear control systems arise in some control engineering theoretical research [Bruzelius, Pettersson and Breitholz, 2004], electrical networks and power generation [García-Planas and Domínguez-García, 2013], light airplanes [Lampton and Valasek, 2012], mechanical systems [Mailybaev, 2003] and automata theory [Pavlov, 2007] among others.

This paper surveys some recent results on feedback classification of linear systems over commutative rings and their meaning in terms of parameter-varying linear systems.

The underlying cornerstone results are of algebraic nature relating a compact topological space Λ with its ring of continuous real functions $\mathcal{C}(\Lambda)$:

Theorem 1.1 (cf. [Atiyah and Macdonald, 1969]). *Compact space Λ is homeomorphic to the maximal spectrum of $\mathcal{C}(\Lambda)$, which is the set of maximal ideals of $\mathcal{C}(\Lambda)$ together with Zariski's topology.*

Theorem 1.2 (Swan, [Rosemberg, 1994]). *Monoid $\mathrm{Vec}_{\mathbb{R}}(\Lambda)$ of finite dimensional vector bundles over Λ is isomorphic to monoid $\mathbf{P}(\mathcal{C}(\Lambda)$ of finitely generated projective $\mathcal{C}(\Lambda)$-modules via global sections.*

These two results will give the bridge connecting recent classification results of algebraic nature [Carriegos, 2013], [Carriegos and Muñoz-Castañeda, 2013], to their geometric interpretation applied to linear parameter-varying systems.

The paper is organized as follows

To conclude the introduction it is worth to remark some recent references dealing with some applications of parameter-varying linear systems [Bruzelius, Pettersson and Breitholz, 2004], [García-Planas and Domínguez-García, 2013], [Lampton and Valasek, 2012], [Mailybaev, 2003], [Pavlov, 2007]

2 Definitions

In the sequel we denote by Λ a compact topological space and by $R = \mathcal{C}(\Lambda)$ the continuous real functions defined on Λ. Note that point wise sum and product gives $\mathcal{C}(\Lambda)$ a commutative ring structure.

Definition 2.1. *A parameter-varying linear system is just a pair of matrices $\Sigma(\lambda) = (A(\lambda), B(\lambda)) \in R^{n\times n} \times R^{n\times m}$. The entries of both matrices are thus continuous real functions defined on Λ.*

Linear system $\Sigma(\lambda) = (A(\lambda), B(\lambda))$ represents the right-hand-side equation

$$\vec{x}^{+}(t) = A(\lambda)\vec{x}(t) + B(\lambda)\vec{u}(t) \tag{1}$$

where $x = \vec{x}$ are the internal states of system, x^{+} is the next-state (shift operator in the discrete framework or time derivative in the continuous case), $u = \vec{u}$ are the external inputs and $\lambda \in \Lambda$ denotes the external parameters of system.

Recall that a linear system $\Sigma(\lambda) = (A(\lambda), B(\lambda))$ is reachable if its reachability matrix

$$A(\lambda)^{*}B(\lambda) = \big(B(\lambda), A(\lambda)B(\lambda), ..., A(\lambda)^{n-1}B(\lambda)\big) \tag{2}$$

represents a surjective linear map. Reachability is a point wise property [Carriegos, Hermida-Alonso and Sánchez-Giralda, 1998], that is to say, a linear system is reachable over $\mathcal{C}(\Lambda)$ if and only if all evaluations are reachable over $\mathbb{R}$.

A feedback action on the linear system $\Sigma(\lambda) = (A(\lambda), B(\lambda))$ is the finite composition of some basis changes in the state-space given by invertible $n \times n$ matrices $P = P(\lambda)$, in the input space given by invertible $m \times m$ matrices $Q = Q(\lambda)$ together with closed loops given by $m \times n$ matrices $F = F(\lambda)$. A general feedback action is thus given by

$$(A, B) \mapsto (P^{-1}AP + P^{-1}BQF, P^{-1}BQ) \qquad (3)$$

Two linear systems are said to be feedback equivalent if one can be transformed into the another by a feedback action. It is not hard to prove that feedback equivalence is not a point wise property.

It is worth to remark here that geometric properties of domain Λ are critical for our purpose of studying feedback actions on linear system $\Sigma(\lambda)$. In the case of linear constant systems over $\mathbb{R}$ (i.e. in our framework, when $\Lambda = \{*\}$ is a singleton) one has two cornerstone results:

1. **Kalman's Decomposition [Kalman, 1972]**: Any linear system splits in a reachable system together with a zero-control system.
2. **Brunovsky's Theorem [Brunovsky, 1970]**: Reachable system (A, B) is characterized (up to feedback actions) by the Kronecker Indices of pencil of matrices

$$(z\mathbf{1} - A, B) \qquad (4)$$

Both results don't hold in the case of linear parameter-varying linear systems.

Let $\Sigma = (A, B) \in \mathbb{R}^{n\times n} \times \mathbb{R}^{n\times m}$ be a constant reachable system. Then one has the following invariants:

1. **Pointwise Invariants** [Carriegos *et al.*, 2013], [Carriegos, Hermida-Alonso and Sánchez-Giralda, 1998]. Determinantal ideals of partial reachability maps [Kalman, 1972]

$$\mathcal{U}_j \left(B, AB, A^2B, ..., A^{i-1}B\right) \qquad (5)$$

2. **Global Invariants**:
   (a) The first global invariants [Kalman, 1972] are the column space N_i^Σ of partial reachability matrices $(B, AB, ..., A^{i-1}B)$.
   (b) The cockernels $M_i^\Sigma = R^n/N_i^\Sigma$ of partial reachability matrices are also feedback invariants [Hermida-Alonso, Pilar Pérez and Sánchez-Giralda, 1996]
   (c) The kernels $I_i^\Sigma = \ker\left(\mathbf{1} : M_{i-1}^\Sigma \to M_i^\Sigma\right)$ [Hermida-Alonso, Pilar Pérez and Sánchez-Giralda, 1996], [Carriegos, 2013].
   (d) The list of feedback invariants includes [Carriegos, 2013] the kernels $Z_i^\Sigma = \ker\left(\bar{A} : I_i^\Sigma \to I_{i+1}^\Sigma\right)$

Definition 2.2. *We say that a linear system $\Sigma(\lambda)$ is quasi-regular if the invariant modules are projective. If system is also reachable then we say that it is regular, regular systems are also called 'locally Brunovsky' systems.*

On the other hand, we say that a linear system is of Brunovsky type or a Brunovsky linear system if it is feedback equivalent to a Brunovsky canonical form.

We also say that a linear system is of Kalman type or is a Kalman system if it is feedback equivalent to a Kalman normal form.

Note that a linear system is regular if all invariants are vector bundles. On the other hand it is well known that every quasi-regular system is of Kalman type [Carriegos and García-Planas, 2004].

3 State Feedback

Now we state the main result of the paper concerning state feedback: Regular systems are given by some kind of decompositions of trivial vector bundle.

Theorem 3.1. *Let $\Sigma = \Sigma(\lambda)$ be a linear time-varying system where parameters $\lambda \in \Lambda$ live in a compact topological space Λ. Then the assignment $\Sigma \mapsto Z(\Sigma) = (Z_i^\Sigma)_{i\geq 1}$ is bijective between the set of regular systems over Λ and the set of solutions of equation*

$$T(n) = z_1 \oplus z_2^2 \oplus \cdots \oplus z_n^n \qquad (6)$$

where $z_i \in \mathrm{Vect}_{\mathbb{R}}(\Lambda)$ are vector bundles over Λ, $T(n)$ is the trivial n-dimensional vector bundle and $\oplus$ is the Whitney sum.

Proof.- Put $R = \mathcal{C}(\Lambda$. Regular linear system Σ is characterized, up to feedback, by its invariants $Z_1^\Sigma, Z_2^\Sigma, ..., Z_n^\Sigma$ (see [Carriegos, 2013]). These invariants are a decomposition of state-space R^n on the form

$$R^n = Z_1^\Sigma \oplus \left(Z_2^\Sigma\right)^2 \oplus \cdots \oplus \left(Z_n^\Sigma\right)^n \qquad (7)$$

Since invariants are projective it follows that above decomposition of state-space R^n translates to a decomposition of trivial vector bundle $T(n)$ over Λ in a Whitney sum of invariant vector bundles by means of global sections.

Corollary 3.2. *On the other hand if Λ is a contractile compact topological space then every vector bundle is (homeomorphic to) trivial and hence searching for solutions of above equation is the same problem to searching for all integer partitions of n. This classical problem was solved by Euler.*

Despite the contractile case, monoid $\mathrm{Vect}_{\mathbb{R}}(\Lambda)$ of vector bundles over Λ is not easily given in general. As an example consider $\Lambda = \mathbb{S}^2$ be the real sphere. then vector bundles are (up to homeomorphisms) [Rosemberg, 1994] elements of the set

$$\mathrm{Vect}_{\mathbb{R}}(\mathbb{S}^2) = \{L_{0,0}, L_{1,0}, L_{2,k}, L_{n,p} : k \in \mathbb{Z}, n \geq 3, p \in \mathbb{Z}_2\} \quad (8)$$

and Whitney sum operates with $L_{0,0}$ as identity element and for $n, m \geq 1$,

$$L_{n,i} \oplus L_{m,j} = L_{n+m,(i+j)\mathrm{mod}2} \quad (9)$$

Above monoid is not easy to manage. In fact there are infinite possible decompositions of trivial bundles $T(n)$. Hence there are infinite regular systems over the sphere.

However it is possible to study some generalizations of feedback equivalence: the dynamic feedback equivalence [Brewer and Klingler, 1988] and the stable feedback equivalence [Carriegos and Muñoz-Castañeda, 2013]. For this kind of equivalence the invariants are collected in the Grothendieck completion $K^0(\Lambda)$ of monoid $\mathrm{Vect}_{\mathbb{R}}(\Lambda)$, which is an easier object to work with. The reader is referred to [Carriegos and Muñoz-Castañeda, 2013] for general reading on this subject.

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