Forced nutations of a two-layer Earth in canonical formulation with dissipative Hori-like kernel

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Abstract

In this research, a modification of the Lie-Hori perturbation method developed by the authors in a recent investigation is used to compute the forced nutations of a non-rigid Earth model, including dissipative processes at the core-mantle boundary. The study is tackled within the Hamiltonian formalism of a two-layer Earth, where the viscous and electromagnetic couplings between mantle and core are introduced via generalized forces. The modified Lie-Hori method is introduced within the framework of the generalized Hamiltonian formalism. It, therefore, allows for calculating first-order perturbations in both conservative and non-conservative systems, while the classical Lie-Hori procedure is not designed to include generalized forces in the kernel to account for dissipative processes. Unlike other methods, ours presents the advantage of keeping the same dimensionality of the original problem, avoiding the doubling of the dimension of the phase space. With this mathematical refinement, differences in the derived nutation amplitudes at the microarcsecond level have been found when compared with the former, first approximation for dissipative systems based on damped oscillators —the only existing previous solution. Those figures are of relevance according to recent recommendations of the International Astronomical Union (IAU) and the International Association of Geodesy (IAG) based on the final report of the Joint Working Group on Theory of Earth rotation and validation.

Keywords:
Perturbation theory, Non-canonical system, Non-linear system, Hamiltonian Mechanics, Earth rotation, Earth Nutation

1. Introduction

Earth's rotation is a fundamental component in transforming Geocentric Celestial Reference System (GCRS) into International Terrestrial Reference System (ITRS) and viceversa. During the last decade the techniques of observation have experienced remarkable improvements, particularly in the precession and nutation parts. Their values are obtained completely from VLBI (very-long baseline interferometry) observations —in the form of offsets of the celestial intermediate pole (CPO). However, a very large part of the variance of the determined CPO still remains unexplained (Malkin [2014]).

CPO being the deviations between the observed celestial intermediate pole and its nominal position (Petit & Luzum [2010]) given by the IAU2000A nutation theory (Mathews et al. [2002]) and the IAU2006 precession theory (Capitaine et al. [2003]), that variance is an indicator of the ability of theories to describe the observations. After many years with no relevant improvement of the unexplained variance, the International Astronomical Union (IAU) and the International Association of Geodesy (IAG) established two successive Joint Working Groups (JWG) to get more insight into the problem, focusing on the potential deficiencies and inaccuracies of theories and the consistency issues that may cause systematic errors. The final report of the IAU/IAG JWG on Theory of Earth rotation and validation (Drewes & Kuglitsch [2019]) led to the approval of Resolution 5 of the 2019 IAG General Assembly, which encouraged a prompt improvement of the Earth rotation theory regarding its accuracy, consistency, and ability to model and predict the essential Earth orientation parameters EOP (Ferrándiz et al. [2020]).

The current IAU2000 nutation model (Mathews et al. [2002]) follows the transfer function approach, whose theoretical background, consisting in finding the ratio of the nutation amplitudes for linear solutions of the rigid and non-rigid cases, was also applied in the previous IAU nutation model, IAU1980 (Seidelmann [1982]). Specifically, IAU2000 relays on the Hamiltonian rigid-Earth solution REN2000 (Souchay et al. [1999]) and uses the transfer function MHB2000 (Mathews et al. [2002]) derived for a symmetric, three-layer Earth, composed of a deformable mantle, a liquid outer core, and a rigid solid inner core —which is a generalization of the two-layer Earth of Sasao et al. ([1980]).

This approach is just one of the possible ones able to tackle the Earth rotation problem as detailed, for example, in Moritz & Mueller ([1987]), Kinoshita & Sasao ([1989]), or Escapa et al. ([2001]). Among them, one remarkable example is the Hamiltonian formalism of the non-rigid Earth initiated by Getino & Ferrándiz. It joins the advantages of the variational methods (e.g., Moritz [1982] or Escapa [2012]) and allows, in contrast to transfer-function approach, the derivation of second-order terms, i.e. a non-linear solution, through Lie-Hori perturbation method. In addition, the Hamiltonian framework is capable of incorporating both precession and nutation motions (e.g., Ferrándiz et al. [2004] or Escapa et al. [2017]), as well as providing analytical nutation formulae. Indeed, the GF2000 three-layer Earth solution, which extended the previous two-layer one by Getino & Ferrándiz ([2001]), played a major role for the validation of the MHB2000 solution before its adoption (Dehant et al. [2002]).

In this paper the modified Lie-Hori perturbation method —from now on MLH method— developed in Baenas et al. ([2017b])—hereinafter Paper I—is applied to the problem of the rotation of a two-layer non-rigid Earth with dissipative effects[1] . This mechanical problem was introduced in Paper I as a relevant example of non-linear mechanical system.

The modified Lie-Hori method is an extension of the perturbation method based on Lie series introduced by Hori ([1966]), for the case of first-order perturbations, with limited applicability to a certain set of dynamical systems. These systems are allowed to have canonical forces in the unperturbed situation (generalized canonical systems), what is the clear advantage over Hori's procedure. By cons, the perturbation terms of the Hamiltonian function must be of an specified form in the dependence with the canonical variables of the problem, as explained in detail in Paper I. However, the procedure is a custom design taking into account the mathematical features of the perturbation terms in the Earth's rotation problem, looking for a rigorous handling of the dissipative effects described via canonical forces, and attending to the restrictions of the method.

Therefore, sections 6.1 and 6.2 of Paper I will be assumed as known, allowing to begin this work from the Hamiltonian equations of the system, written down in a canonical set of non-singular variables. On the features of this canonical set Getino at al. ([2000]) and Escapa et al. ([2001]) can be consulted. Non-singular variables are commonly used in Celestial Mechanics, both for orbital motion (e.g., Poincaré variables, see Brouwer & Clemence [1961]) and rotational motions (e.g., Fukushima [1994] or Lara [2014]).

As stated in Paper I, the use of a canonical set of this kind instead of the usual one of Andoyer-like variables for the two-layer Earth (Getino [1995], Getino & Ferrándiz [2001]) allows to meet mathematical requirements for the application of the modified Lie-Hori perturbation method in order to take advantage of an Hori auxiliary system defined by a linear system of first-order differential equations with constant coefficients.

The main purpose of this work is to tackle the dissipative effects at the core-mantle boundary (CMB) with a rigorous procedure within the Hamiltonian formalism of the Earth rotation, in order to obtain the forced nutations of the figure axis. The dissipative effects in the CMB were firstly incorporated to the Hamiltonian theory of a two-layer Earth by Getino & Ferrándiz ([1997], [2001]), by writing the viscous and electromagnetic couplings through the generalized forces approach. That theoretical framework took advantage of the small magnitude of the dissipation at the CMB to assume a numerical simplification consisting in the application of the Lie-Hori algorithm for Hamiltonian systems, to generalized canonical ones that are based on damped oscillators. Such approximation implies the introduction of an ad-hoc negative sign for the damping constant, as was shown in Paper I (example I) when studying the driven damped harmonic oscillator. In this work, the accuracy of such approximate Lie-Hori procedure—referred to with the acronym ALH—will be checked in the scope of a much more complex problem as the non-rigid Earth rotation Hamiltonian theory.

The structure of this paper is the following. In Sect. 2 the lunisolar-driven two-layer Earth rotational problem, including dissipation at the CMB, is succinctly introduced and formulated in terms of an appropriate canonical set for the use of the MLH method within the Hamiltonian approach. Some mathematical arrangements—based on the known ALH procedure—are performed throughout Sect. 3 in order to simplify the application of the perturbative equations in Sect. 4. Then, Sect. 5 shows the derived analytical formulae describing the forced nutations of the Earth (terms sensitive to the dissipation effects), these new formulae being the main contribution of this research to the Hamiltonian theory of the Earth rotation. Finally, in Sect. 6 the related nutational components are numerically calculated, allowing the comparison between ALH and MLH methods in quantitative terms. This implies a numerical validation of the approximate method, not directly carried out to date.

2. Formulation of the problem in non-singular variables

The Hamiltonian framework of a non-rigid Earth including a dissipative coupling torque at the CMB can be consulted in Getino et al. ([2000]). The problem is formulated in the non-singular variables set (ibid)

$$\{\lambda, y_1, y_2, \Lambda, Y_1, Y_2, \lambda_c, y_{c1}, y_{c2}, \Lambda_c, Y_{c1}, Y_{c2}\}, \tag{1}$$

where the variables with no subscript belong to the whole Earth, while c subscript does it for the fluid outer core (FOC). The rotational dynamics is derived from the application of the generalized Hamiltonian equations, which can be cast in matrix form as

$$\frac{dQ}{dt} = -E_4 \nabla_Q \mathcal{H} + E_4 Q, \tag{2}$$

$\mathcal{H}$ being the Hamiltonian of the system, E_4 the 4-dimensional symplectic matrix, and Q a vector comprising the canonical forces (the Appendix A of Paper I can be consulted for further details).

The (y, Y) pairs are similar to the well-known Poincaré variables used to remove virtual singularities, and constitute a suitable canonical set for the Hori auxiliary system—or Hori kernel—to be written down through a linear system of first-order differential equations with constant coefficients, namely,

$$\frac{dQ}{dt} = MQ, \tag{3}$$

[1] In spite of the need of using of a three-layer Earth model for the main effects, when dealing with perturbations of small magnitude the most convenient way is using a two-layer Earth model (e.g., Lambert & Mathews ([2006]), Baenas et al. [2017a], [2019], etc.). As described in Paper I, MLH is of general application for certain kind dynamical systems, including one-, two-, and three-layer Earth models

M being a constant matrix, and $Q^T = (Y_2, Y_{c2}, y_2, y_{c2})$.

The Hori kernel refers to the unperturbed situation of the system, denoted as UP, and in this case includes the generalized forces describing the dissipation at the CMB. The four variables appearing in Eq. (3) are coupled through the differential system, while the remaining ones are uncoupled. The solution of the uncoupled variables related to the whole Earth has the form $\lambda = \lambda_0$, $\Lambda = \Lambda_0$, $Y_1 = Y_{1,0}$, $y_1 = \omega_E t + y_{1,0}$; 0 subscript denotes initial values, and ω_E the mean Earth's angular velocity.

The M matrix can be obtained from Getino et al. ([2000]), when reducing their triaxial model to a symmetrical one, and is

$$M = \begin{pmatrix} 0 & 0 & -m_1 & m_2 \\ -m_2 S & -m_3 S & m_2(1+S') & -m_3(1+S') \\ m_1 & m_2 & 0 & 0 \\ m_2(1+S') & m_3(1+S') & m_2 S & -m_3 S \end{pmatrix}, \tag{4}$$

where m_i constants depend on the principal inertia moments of the Earth[2], while S and S' constants come from the viscous and electromagnetic dissipative torques, respectively (and thus vanish in the non dissipative case[3]).

In the main problem of a two-layer deformable Earth tackled by Getino & Ferrándiz ([2001]), the perturbation is split into the lunisolar tidal (or tide-raising) potential energy of second harmonic degree, $\mathcal{V}_0$, and the mass redistribution kinetic energy, $\mathcal{T}_t$. In this work, only the Oppolzer terms—quasi-periodic components of the motion of the Earth's figure axis with respect to the angular momentum one—of the nutations will be considered, since there is no influence of the dissipative effects on the Poisson terms—quasi-periodic components of the motion of the Earth's angular momentum axis—at this order of perturbation. Therefore, the first-order Hamiltonian can be reduced to their tesseral part, namely,

$$\begin{aligned}
\mathcal{V}_0 &= -k' \sum_{i,\tau=\pm1} C_{i,\tau} \cos(\tau\Theta_i - y_1) \frac{Y_2}{\sqrt{Y_1}} \\
&\quad -k' \sum_{i,\tau=\pm1} C_{i,\tau} \sin(\tau\Theta_i - y_1) \frac{y_2}{\sqrt{Y_1}}, \\
\mathcal{T}_t &= \sum_{i,\tau=\pm1} C_{i,\tau} \left\{ k_t^m \begin{bmatrix} \sqrt{Y_1} Y_2 \cos(n_{i,\tau} t + n_{i,0}) \\ -\sqrt{Y_1} y_2 \sin(n_{i,\tau} t + n_{i,0}) \end{bmatrix} \right. \\
&\quad \left. + (k_t^m - k_t^c) \begin{bmatrix} \sqrt{Y_{c1}} Y_{c2} \cos(n_{i,\tau} t + n_{i,0}) \\ + \sqrt{Y_{c1}} y_{c2} \sin(n_{i,\tau} t + n_{i,0}) \end{bmatrix} \right\}.
\end{aligned} \tag{5}$$

Here, $C_{i,\tau}$ functions are trigonometric polynomials in $\cos^{-1}\left(Y_1^{-1}\Lambda\right)$ (the argument is hidden for the sake of brevity), k' is a constant related to the gravitational interaction, while $k_t^{m,c}$ are elastic constants for mantle and core, respectively (Getino & Ferrándiz [2001]). As usual, the fundamental arguments Θ_i are linear combinations of the Delaunay variables of the Moon and the Sun (see Kinoshita [1977] for details). Finally, we have $n_{i,\tau} = \omega_E - \tau n_i$, n_i being the i-th orbital frequency related to the argument Θ_i.

It should be noted that the restriction $\mathcal{H}_{1UP}$ of the perturbation $\mathcal{H}_1 = \mathcal{V}_0 + \mathcal{T}_t$ to the auxiliary system is linear in the coupled variables, and thus meets mathematical requirements of the MLH method. The algorithm requires replacing the auxiliary system by a new one, through the transformation $M \to M^*$ given by $M^* = E_4 M^T E_4$, E_4 being the 4-dimensional symplectic matrix. Thus,

$$M^* = \begin{pmatrix} 0 & -m_2 S & -m_1 & m_2(1+S') \\ 0 & -m_3 S & m_2 & -m_3(1+S') \\ m_1 & m_2(1+S') & 0 & m_2 S \\ m_2 & m_3(1+S') & 0 & m_3 S \end{pmatrix}, \tag{6}$$

is obtained. The new Hori kernel, denoted as UP^*, is defined by Eq. (3), with matrix M^* instead of M. In the non dissipative case, $M^* = M$ (Paper I, Proposition 1).

The first-order generating function of the MLH method is given by the integral

$$\mathcal{W}_1 = \int_{UP^*} \mathcal{H}_1 dt. \tag{7}$$

It can be performed by a procedure similar to that sketched in Paper I. Due to the linearity of $\mathcal{H}_{1UP^*}$ in the Q variables, the integrand of Eq. (7) can be written in the form $Qe^{ih_{i,\tau}}$ with $h_{i,\tau} = y_1 - \tau\Theta_i = n_{i,\tau}t + n_{i,0}$. Hence, taking into account that the solution of the auxiliary system UP^* can be cast as $Q = e^{M^* t} Q_0$, the generating function is arranged in terms of integrals of the type

$$\int_{UP^*} Qe^{ih_{i,\tau}} dt = \left(M^* + i n_{i,\tau} I_4\right)^{-1} Qe^{ih_{i,\tau}}, \tag{8}$$

I_4 being the 4-dimensional identity matrix.

3. Formulation of the problem in complex variables

The non-singular canonical set has allowed the application of the MLH method and the writing of the integral defining the generating function in a suitable manner. However, once the integration is performed, it is convenient to transform the equations of the problem into the following set of complex variables, based on the Andoyer's canonical set,

$$u = iM_E \sin\sigma e^{-i\nu}, \; v = -iM_c \sin\sigma_c e^{i\nu_c}, \tag{9}$$

and their complex conjugated ones, $\bar{u}$ and $\bar{v}$ (Getino & Ferrándiz [2001]). In Eq. (9), M_E is the modulus of the angular momentum vector of the whole Earth, and M_c that of the FOC, while σ and σ_c are auxiliary angles defined by $N_E = M_E \cos\sigma$ and $N_c = M_c \cos\sigma_c$, N_E and N_c being the projections of the angular momentum of the Earth and the core with respect to the zeta axis of the terrestrial reference system. In turn, ν and ν_c are Andoyer's coordinates related to the polhode (Getino [1995]).

Although the transformation from Q variables to the new ones $U^T = (u, v, \bar{u}, \bar{v})$ is not canonical, this set allows an easier construction of analytical expressions. The transformation

[2] In what follows, A, A_m, A_c and C, C_m, C_c stand for the equatorial and polar principal moments of the axial symmetric Earth, mantle, and core, respectively.

[3] This torque also vanishes when the equatorial components of the angular velocities of the mantle and the core are equal, as can be derived from Eq. (3) and Getino et al. ([2000])

between U and Q is given by $U = M_U Q$, M_U matrix being

$$M_U = \begin{pmatrix} i\sqrt{Y_1} & 0 & -\sqrt{Y_1} & 0 \\ 0 & -i\sqrt{Y_{c1}} & 0 & -\sqrt{Y_{c1}} \\ -i\sqrt{Y_1} & 0 & -\sqrt{Y_1} & 0 \\ 0 & i\sqrt{Y_{c1}} & 0 & -\sqrt{Y_{c1}} \end{pmatrix}. \qquad (10)$$

As $Y_1 = C\omega_E$ and $Y_{c1} = C_c\omega_E$, in the auxiliary system UP^*, M_U becomes a constant matrix.

Combining Eqs. (8) and (10), we obtain

$$\int_{UP^*} U e^{ih_{i,\tau}} dt = \left(R_4^* + in_{i,\tau} I_4\right)^{-1} U e^{ih_{i,\tau}}, \qquad (11)$$

R_4^* being defined by the transformation of M^* under M_U, i.e.,

$$R_4^* = M_U M^* M_U^{-1}. \qquad (12)$$

This matrix has a diagonal block structure, given by a 2-dimensional matrix, R_2^*, and its complex conjugated, $\overline{R_2^*}$,

$$R_4^* = \begin{pmatrix} R_2^* & 0_2 \\ 0_2 & \overline{R_2^*} \end{pmatrix}, \qquad (13)$$

with 0_2 being the 2-dimensional null matrix. Thus, Eq. (11) can be rewritten in the following form, where the dimension of the involved matrices is reduced to 2,

$$\int_{UP^*} \begin{pmatrix} u \\ v \end{pmatrix} e^{ih_{i,\tau}} dt = \left(R_2^* + in_{i,\tau} I_2\right)^{-1} \begin{pmatrix} u \\ v \end{pmatrix} e^{ih_{i,\tau}}. \qquad (14)$$

Matrix R_2^* is obtained after some algebra From Eq. (12). It can be written down in a suitable form as

$$R_2^* = i\tilde{R}_2^*, \ \tilde{R}_2^* = \begin{pmatrix} r_1^* & r_2^* - i\omega_E \frac{C}{C_c}\Gamma \\ r_3^* & r_4^* - i\omega_E \frac{A}{A_c}\Gamma \end{pmatrix}, \qquad (15)$$

in terms of $\tilde{R}_2^*$ matrix, by the sake of algebraic convenience. Its elements are

$$\begin{aligned} r_1^* &= \omega_E \frac{C-A}{A_m} = \omega_E \left(\frac{A_c}{A_m} + e\frac{A}{A_m}\right), \\ r_2^* &= -\omega_E \frac{C}{A_m}(1+S') = -\omega_E \left[\frac{A}{A_m}(1+e) + \frac{C}{C_c}\Gamma'\right], \\ r_3^* &= \omega_E \frac{C_c}{A_m} = \omega_E \frac{A_c}{A_m}(1+e_c), \\ r_4^* &= -\omega_E \frac{C_c}{A_c}\frac{A}{A_m}(1+S') = -\omega_E \left[\frac{A}{A_m}(1+e_c) + \frac{A}{A_c}\Gamma'\right], \end{aligned} \qquad (16)$$

Γ and Γ' being the dimensionless dissipative constants (from viscous and electromagnetic coupling, respectively) in the notation by Getino & Ferrándiz ([2001]), and e and e_c the Earth and core ellipticities, namely,

$$\begin{aligned} \Gamma &= \frac{C_c}{A_m} S, \ \Gamma' = \frac{C_c}{A_m} S', \\ e &= \frac{C-A}{A}, \ e_c = \frac{C_c - A_c}{A_c}. \end{aligned} \qquad (17)$$

4. Amplitude functions for the nutations

The calculation of the integral given in Eq. (14) is required in order to obtain closed-form analytical expressions for the nutation amplitudes. We will use the eigenvalues of $\tilde{R}_2^*$ to compute the inverse of $R_2^* + in_{i,\tau} I_2$. These are given by

$$\lambda_1 = m_1, \ \lambda_2 = m_2 + id, \qquad (18)$$

where

$$\begin{aligned} m_1 &= \omega_E \frac{A}{A_m} e, \\ m_2 &= -\omega_E \left(1 + \frac{A}{A_m} e_c + \frac{A}{A_c}\Gamma'\right), \\ d &= -\omega_E \Gamma \frac{A}{A_c}. \end{aligned} \qquad (19)$$

The comparison of these parameters with those of GF2001 (Sect. 5.1) for the unperturbed solutions, shows that the dissipation parameter d has opposite sign. This is not really a discrepancy, since the former parameters only affects the calculation of the generating function of the problem—which is related to the perturbed situation and the Hori kernel—but not the unperturbed problem (UP), whose eigenvectors correspond to the Earth's free frequencies—Chandler wobble (CW) and free core nutation (FCN).

Therefore, the inverse of $A = R_2^* + in_{i,\tau} I_2$ can be calculated as

$$\begin{aligned} A^{-1} &= \frac{\mathrm{Adj}\left(\tilde{R}_2^* + n_{i,\tau} I_2\right)}{\det\left(\tilde{R}_2^* + n_{i,\tau} I_2\right)} = \frac{1}{f_1\,(f_2 + id)} \\ &\times \begin{pmatrix} n_{i,\tau} + r_4^* - i\omega_E \frac{A}{A_c}\Gamma & -r_2^* + i\omega_E \frac{C}{C_c}\Gamma \\ -r_3^* & n_{i,\tau} + r_1^* \end{pmatrix}, \end{aligned} \qquad (20)$$

with

$$\begin{aligned} f_1 &= m_1 + n_{i,\tau} = m_1 + \omega_E - \tau n_i, \\ f_2 &= m_2 + n_{i,\tau} = m_2 + \omega_E - \tau n_i. \end{aligned} \qquad (21)$$

Finally, Eq. (14) can be decomposed in two auxiliary integrals,

$$\begin{aligned} I_1 &= \mathrm{Im}\left\{\int u e^{ih_{i,\tau}} dt\right\} = \int M \sin\sigma \cos\left(h_{i,\tau} - \nu\right) dt, \\ I_2 &= \mathrm{Im}\left\{-\int v e^{ih_{i,\tau}} dt\right\} = \int M_c \sin\sigma_c \cos\left(h_{i,\tau} + \nu_c\right) dt, \end{aligned} \qquad (22)$$

which can be obtained using Eqs. (20) and (14) in the form,

$$\begin{aligned} I_1 &= M_E \sin\sigma \left[F_1^a \sin(h-\nu) + F_1^b \cos(h-\nu)\right] \\ &\quad + M_c \sin\sigma_c \left[F_2^a \sin(h+\nu_c) + F_2^b \cos(h+\nu_c)\right], \\ I_2 &= M_E \sin\sigma \left[G_1^a \sin(h-\nu) + G_1^b \cos(h-\nu)\right] \\ &\quad + M_c \sin\sigma_c \left[G_2^a \sin(h+\nu_c) + G_2^b \cos(h+\nu_c)\right], \end{aligned} \qquad (23)$$

by means of the definition of a set of amplitude functions—denoted as F and G—that are given by

$$
\begin{aligned}
F_1^a &= \frac{f_2\left(n_{i,\tau}+r_4^*\right)}{f_1\left(f_2^2+d^2\right)}, \qquad & F_1^b &= -\Gamma\omega_E\frac{A}{A_c}\frac{f_2-r_4^*-n_{i,\tau}}{f_1\left(f_2^2+d^2\right)},\\
G_1^a &= \frac{r_3^* f_2}{f_1\left(f_2^2+d^2\right)}, & G_1^b &= \Gamma\omega_E\frac{A}{A_c}\frac{r_3^*}{f_1\left(f_2^2+d^2\right)},\\
F_2^a &= \frac{r_2^* f_2}{f_1\left(f_2^2+d^2\right)}, & F_2^b &= \Gamma\omega_E\frac{\frac{A}{A_c}r_2^*-\frac{C}{C_c}f_2}{f_1\left(f_2^2+d^2\right)},\\
G_2^a &= \frac{f_2\left(n_{i,\tau}+r_1^*\right)}{f_1\left(f_2^2+d^2\right)}, & G_2^b &= \Gamma\omega_E\frac{A}{A_c}\frac{n_{i,\tau}+r_1^*}{f_1\left(f_2^2+d^2\right)}.
\end{aligned}
\tag{24}
$$

These functions gather the well-known resonance features that amplify the nutation components with frequencies near to the CW and FCN. These resonances are not pure due to the nonzero value of d parameter.

An identical mathematical procedure can be followed to obtain the amplitude functions corresponding to the ALH method, i.e., the approximation comprising the application of the Lie-Hori method to generalized Hamiltonian systems based on damped oscillators. From a practical point of view, the construction of the ALH procedure only differs from the MLH one in the use of the UP kernel instead of UP^*, and a subsequent replacement of Γ by $-\Gamma$ in the amplitude functions with b superscript. Therefore, the calculation will not be repeated here for the sake of brevity, but the related functions and parameters will be explicitly given—denoted with ALH subscript—for comparative purposes when necessary.

5. Oppolzer terms of the nutations

As previously mentioned, only the Oppolzer terms are sensitive to the dissipation effects at the CMB at the first order of perturbation. Following Paper I, the calculation of the generating function is affected for the LHM method, but the first-order perturbative equations remain unchanged. Therefore, the formulae to obtain the Oppolzer terms are the same used by Getino & Ferrándiz ([2001]), among others.

The Oppolzer terms are $\Delta\left(\lambda_f-\lambda\right)$ and $\Delta\left(I_f-I\right)$ for longitude and obliquity, respectively—where λ_f, I_f, denote the longitude and inclination of the Earth's equator plane (Euler angles), while λ, I, the longitude and inclination of the plane normal to the angular momentum axis (Andoyer angles). In summary, the nutation formulae are

$$
\begin{aligned}
\Delta_0\left(\lambda_f-\lambda\right) &= \frac{k_0}{\sin I}\sum_{i,\tau=\pm1} C_{i,\tau}(I)\begin{bmatrix} F_1^a\tau\sin\Theta_i \\ -F_1^b\cos\Theta_i\end{bmatrix},\\
\Delta_0\left(I_f-I\right) &= k_0\sum_{i,\tau=\pm1} C_{i,\tau}(I)\begin{bmatrix} F_1^a\cos\Theta_i \\ +F_1^b\tau\sin\Theta_i\end{bmatrix},
\end{aligned}
\tag{25}
$$

for the terms arising from the tidal potential energy, $\mathcal{V}_0$, and

$$
\begin{aligned}
\Delta_t\left(\lambda_f-\lambda\right) &= \frac{1}{\sin I}\sum_{i,\tau=\pm1} C_{i,\tau}(I)\begin{bmatrix} E_1\tau\sin\Theta_i \\ -E_2\cos\Theta_i\end{bmatrix},\\
\Delta_t\left(I_f-I\right) &= \sum_{i,\tau=\pm1} C_{i,\tau}(I)\begin{bmatrix} E_1\cos\Theta_i \\ +E_2\tau\sin\Theta_i\end{bmatrix},
\end{aligned}
\tag{26}
$$

with

$$
\begin{aligned}
E_1 &= k_t^m\left[-\cos\epsilon_i\left(F_1^a+G_1^a\right)+\sin\epsilon_i\left(F_1^b+G_1^b\right)\right]\\
&\quad +k_t^c\left(\cos\epsilon_i G_1^a-\sin\epsilon_i G_1^b\right),\\
E_2 &= k_t^m\left[-\cos\epsilon_i\left(F_1^b+G_1^b\right)-\sin\epsilon_i\left(F_1^a+G_1^a\right)\right]\\
&\quad +k_t^c\left(\cos\epsilon_i G_1^b+\sin\epsilon_i G_1^a\right),
\end{aligned}
\tag{27}
$$

for the terms coming from the mass redistribution kinetic energy, $\mathcal{T}_t$. Here, ϵ_i is an ad-hoc phase lag angle introduced in the arguments of the nutations to take into account the anelastic response of the mantle.

Although this work is just focused on the evolution of the Earth's figure axis, one of the advantages of the Hamiltonian approach is that it allows to obtain the evolution of any other function of the canonical variables in a systematic way. This is the case, for example, of the components of the angular velocity of the mantle in the Earth reference system. They can be tackled in an analogous way to the rigid case (Kinoshita [1977]), adapting the corresponding formulation to the loss of osculation of Andoyer variables (Efroimsky & Escapa [2007], Escapa[2011]).

6. Numerical results and discussion

The Oppolzer terms due to the $\mathcal{V}_0$ potential only depend on the F_1^a and F_1^b amplitude functions. These functions are identical to those of the ALH method.

The situation is somehow different for the nutations arising from the $\mathcal{T}_t$ Hamiltonian. First, although G_1^a and $G_{1,\mathrm{ALH}}^a$ seem to be formally identical, there exists an implicit difference because $r_3^*\neq r_{3,\mathrm{ALH}}$, namely,

$$
\begin{aligned}
r_3^* &= \omega_E\frac{A_c}{A_m}(1+e_c),\\
r_{3,\mathrm{ALH}} &= \omega_E\left[\frac{A_c}{A_m}(1+e_c)+\Gamma'\right].
\end{aligned}
\tag{28}
$$

This difference correspond to the electromagnetic coupling constant, and taking into account the magnitude of the involved parameters—$\Gamma'\sim10^{-6}$, $(A_c/A_m)(1+e_c)\sim10^{-1}$, see, e.g., Getino & Ferrándiz ([1997], [2000])—the discrepancy has no numerical relevance, i.e., $r_3^*\simeq r_{3,\mathrm{ALH}}$, and therefore, $G_1^a\simeq G_{1,\mathrm{ALH}}^a$.

The main difference between the two cases under analysis arises from G_1^b function, since $G_1^b\neq G_{1,\mathrm{ALH}}^b$, namely,

$$
\begin{aligned}
G_1^b &= \Gamma\omega_E\frac{A}{A_c}\frac{r_3^*}{f_1\left(f_2^2+d^2\right)},\\
G_{1,\mathrm{ALH}}^b &= -(-\Gamma)\,\omega_E\frac{\frac{A}{A_c}r_3^*+f_2}{f_1\left(f_2^2+d^2\right)}.
\end{aligned}
\tag{29}
$$

Table 1: Oppolzer terms of the nutations.

						MLH method		ALH method ($-\Gamma < 0$)			
Argument, Θ_i					Period	Longitude	Obliquity	Longitude	Obliquity	Dif-Lon	Dif-Obl
l	l'	F	D	Ω	(days)	(μas)	(μas)	(μas)	(μas)	(μas)	(μas)
In-phase terms											
0	0	0	0	1	−6793.48	114627.12	−35189.75	114627.08	−35189.74	0.04	−0.01
0	0	0	0	2	−3396.74	−2414.87	898.47	−2414.87	898.47	0.00	0.00
0	1	0	0	0	365.26	32350.86	10838.43	32350.84	10838.43	0.02	0.00
0	−1	2	−2	2	365.25	449.23	−539.70	449.23	−539.70	0.00	0.00
0	0	2	−2	2	182.63	−63509.47	30946.51	−63509.45	30946.50	−0.02	0.01
0	1	2	−2	2	121.75	−2878.83	1321.46	−2878.83	1321.46	0.00	0.00
1	0	0	0	0	27.55	5006.11	−190.20	5006.11	−190.20	0.00	0.00
0	0	2	0	2	13.66	−20103.53	8387.92	−20103.53	8387.91	0.00	0.01
0	0	2	0	1	13.63	−3564.64	1667.70	−3564.64	1667.70	0.00	0.00
1	0	2	0	2	9.13	−2993.81	1230.96	−2993.81	1230.96	0.00	0.00
Out-of-phase terms											
0	0	0	0	1	−6793.48	−263.88	−160.59	−265.57	−161.46	1.69	0.87
0	0	0	0	2	−3396.74	4.27	2.08	4.31	2.10	−0.04	−0.02
0	1	0	0	0	365.26	3235.18	−1313.45	3235.70	−1313.70	−0.52	0.25
0	−1	2	−2	2	365.25	−61.59	15.10	−61.61	15.09	0.02	0.01
0	0	2	−2	2	182.63	1337.38	347.00	1338.84	347.47	−1.46	−0.47
0	1	2	−2	2	121.75	58.19	18.37	58.25	18.39	−0.06	−0.02
1	0	0	0	0	27.55	−0.89	−40.74	−0.89	−40.78	0.00	0.04
0	0	2	0	2	13.66	393.14	147.53	393.46	147.65	−0.32	−0.12
0	0	2	0	1	13.63	79.04	25.79	79.11	25.81	−0.07	−0.02
1	0	2	0	2	9.13	55.85	21.28	55.89	21.30	−0.04	−0.02

Here, the change of sign of Γ does not correct the amplitude difference in absolute value. Since the discrepancy affects to a b function, it will affect mainly to the out-of-phase terms[4].

Table 1 shows the in-phase and out-of-phase components of the Oppolzer terms of a reduced set of the most significant arguments Θ_i calculated according to the two analyzed procedures. The first block of amplitudes corresponds to this work's results, the MLH method, i.e., the nutation formulae (25) and (26), with the amplitude functions given by Eqs. (24). The second is obtained by the ALH procedure, which includes a negative damping constant $-\Gamma$. The computations have been performed using the numerical values in Table 1 of Getino & Ferrándiz ([2001]) for constants and parameters.

It can be appreciated that the difference in the in-phase amplitudes between both cases is at the level of the 10^{-2} μas or less. However, differences at the μas level has been found in the out-of-phase amplitudes, corresponding to the 6798.48-d (18.6-y) and 182.63-d (0.5-y) nutation components.

The magnitudes are below the present uncertainty of the determinations of individual amplitudes that is 2-3 μas (Belda et al. [2017], Gattano et al. [2017]), but even though our results are relevant according to recent recommendations of the International Astronomical Union (IAU) and the International Association of Geodesy (IAG) made through its Joint Working Group on Theory of Earth rotation (Ferrándiz and Gross [2016], Ferrándiz et al. [2020]). They prompt the estimation of the impact on accuracy of the many approximations made in theories, besides urging an improvement of their consistency (see, e.g., Escapa et al. [2017]).

Our results provide an estimation and validation of the use of the ALH approximation within the Hamiltonian approach to the Earth rotation. Besides, they show that future improvements of consistency are reachable with little effort due to the availability of the MLH method.

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[4] In Eqs. (25) and (26), the in-phase terms refer to the nutation components proportional to $\sin\Theta_i$ in longitude and $\cos\Theta_i$ in obliquity. In turn, the out-of-phase terms are those proportional to $\cos\Theta_i$ in longitude and $\sin\Theta_i$ in obliquity.

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