



Research Article

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On the regulator problem for linear systems over rings and algebras

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Abstract: The regulator problem is solvable for a linear dynamical system Σ if and only if Σ is both pole assignable and state estimable. In this case, Σ is a canonical system (i.e., reachable and observable). When the ring R is a field or a Noetherian total ring of fractions the converse is true. Commutative rings which have the property that the regulator problem is solvable for every canonical system (RP-rings) are characterized as the class of rings where every observable system is state estimable (SE-rings), and this class is shown to be equal to the class of rings where every reachable system is pole-assignable (PA-rings) and the dual of a canonical system is also canonical (DP-rings).

Keywords: linear systems over commutative rings, regulator problem, duality principle, pole assignment

MSC 2020: 13P25, 93B25

1 Introduction

The mathematical theory of systems is a theory about mathematical models of real-life systems, which are often given by functions that describe the time dependence of a point in some state-space. This approach assumes that there are several internal variables to the system and that system itself evolves in terms of external inputs together with current internal states. Systems are not always simple enough to be described using linear models, but linear systems are very useful and widely applied. A linear system can then be presented in terms of the time-variable t as the right-hand-side equation together with a read-out equation

$$\begin{aligned}x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t),\end{aligned}\tag{1}$$

where $u(t)$ is the sequence of external inputs to the system, $y(t)$ is the sequence of observations, and $x(t)$ is the sequence of internal states. Inputs and outputs live in finite dimensional real vector spaces in the classical framework. This is also the case of I/O systems realized in a finite dimensional real vector space of internal states. Note that once bases in finite dimensional vector spaces of inputs, states, and outputs are fixed, the above dynamical equation gives rise to a system given by the triple $(A, B, C) \in \mathbb{R}^{n \times n} \times \mathbb{R}^{n \times m} \times \mathbb{R}^{p \times n}$ of real matrices.

In more general cases [1,2], scalars live in a commutative ring R . This commutative ring would be the field \mathbb{C} of complex numbers [3], or in a finite field [4], in the ring $\mathbb{Z}/m\mathbb{Z}$ of modular integers [5], in a polynomial ring [6], or in a ring of continuous functions [7]. In this paper, linear systems are given by triples $(A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$ of matrices with entries in some commutative ring R of scalars. For general reading on the subject of linear systems over commutative rings and its applications the reader is referred to [1,2].

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Some recent studies on the topic are [4,5,7–14]. References [6,15–17] also deal with linear dynamical systems over commutative rings from the feedback regulation point of view.

This paper deals with the study of regulation of linear systems over commutative rings by means of dynamic compensators. The paper is organized as follows: feedback control of linear systems as well as output rejections and dynamic compensator actions are reviewed in Section 2.

Note that if \mathbb{K} is any field, then it is well known that a triple $(A, B, C) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m} \times \mathbb{K}^{p \times n}$ admits a dynamic compensator if and only if (a) pair $(A, B) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{n \times m}$ is reachable; and (b) pair $(A, C) \in \mathbb{K}^{n \times n} \times \mathbb{K}^{p \times n}$ is observable; that is to say, triple (A, B, C) is a canonical one. But however that characterization does not extend to the case of R being an arbitrary commutative ring: the condition of being canonical does not assure, in general, that a dynamic compensator does exist. Section 3 is devoted to review reachability, observability, as well as necessary conditions on a triple, like duality and pole-assignment, in order to admit a dynamic compensator in the case of scalars in a commutative ring R .

DP-rings and duality are reviewed in Section 4. Duality rings (or DP-rings, see [16]), pole-assignability, and PA-rings (see [1,6,11,18], among many others) are related to regulator property.

Next, Section 5 is devoted to solve the regulator property for single-input single-output (SISO) triples $(A, \underline{b}, \underline{c}^t) \in R^{n \times n} \times R^{n \times 1} \times R^{1 \times n}$. To be concise, the class of rings where every SISO canonical triple admits a dynamic compensator is the class of total rings of fractions.

Section 6 deals with the general case of multi-input multi-output linear systems ($m, p \geq 1$). The class of SE-rings is introduced as the class of commutative rings where every observable pair is state-estimable. This latter class of rings is exactly the same as the class of rings where every canonical triple admits a dynamic compensator (RP-rings). Hence, the characterization result

$$\text{RP-rings} = \text{SE-rings}$$

is proved. Even more, the class RP (or SE) is related to former classical classes of PA-rings and DP-rings. To be concise, we prove the following result:

$$\text{RP-rings} = \text{SE-rings} = (\text{PA-rings}) \cap (\text{DP-rings}).$$

Some examples to distinguish several classes of rings related to the problem are described in Section 7; special attention is paid to finite commutative rings as a class of RP-rings useful by their applications, see [4,5,19,20], or [21]. Several examples are also given to circumscribe the class of RP-rings. Finally, conclusion of the paper is given in Section 8.

2 Feedback control, output rejection, and regulation

Feedback action on a linear system

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t) \end{aligned} \tag{2}$$

or closed loop by matrix $K \in R^{m \times n}$ is performed by choosing the inputs $u(t) = v(t) + Kx(t)$ as a linear function of current states. Then system (2) is transformed into

$$\begin{aligned} x(t+1) &= (A + BK)x(t) + Bv(t), \\ y(t) &= Cx(t), \end{aligned} \tag{3}$$

that is to say, triple (A, B, C) is brought to a new triple $(A + BK, B, C)$, which is *equivalent* to the latter in the sense that one of them is obtained from the other by the feedback loop defined by K (Figure 1).

Note that characteristic polynomial of A is not equal to characteristic polynomial of $A + BK$. Hence, feedback loops modify the eigen-structure of state-transition matrix of linear system, and therefore feedback loops modify the behavior of systems.

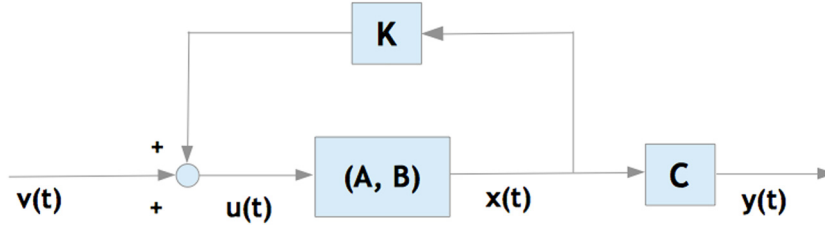


Figure 1: Feedback action $K \in R^{n \times m}$ on linear system $(A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$.

But, on the other hand, measuring internal states $x(t)$ in order to define the feedback loop is in general a hard task because too many sensors are often needed. It would be impossible to access internal states in many real-world cases. Thus, an estimator $\xi(t)$ is often used instead current internal state $x(t)$ to close the feedback loop. Hence, a dynamic compensator scheme is performed as follows in order to give rise to a useful observer $\xi(t)$.

Given the linear system (2), consider the scheme of Figure 2 and the so-called state observer $\xi(t) \in R^n$. This observer is given by the following right-hand side equation:

$$\xi(t + 1) = A\xi(t) + (B, L) \begin{pmatrix} u(t) \\ y(t) - C\xi(t) \end{pmatrix}. \tag{4}$$

Now set the inputs as $u(t) = K\xi(t) + v(t)$. Then dynamics of the observer are as follows:

$$\xi(t + 1) = A\xi(t) + Bu(t) + Ly(t) - LC\xi(t) = A\xi(t) + Bu(t) + LC(x(t) - \xi(t)). \tag{5}$$

The error estimate $\varepsilon(t) = x(t) - \xi(t)$ satisfies the dynamic equation

$$\varepsilon(t + 1) = (A - LC)\varepsilon(t). \tag{6}$$

The above discussion proves, in the context of commutative rings, the following result:

Theorem 2.1. *Let R be a commutative ring. Consider the linear system (1) with scalars in R . Then the dynamic compensator $(K, L) \in R^{m \times n} \times R^{n \times p}$ yields the following linear control system:*

$$\begin{pmatrix} x(t + 1) \\ \varepsilon(t + 1) \end{pmatrix} = \begin{pmatrix} A + BK & -BK \\ 0 & A - LC \end{pmatrix} \begin{pmatrix} x(t) \\ \varepsilon(t) \end{pmatrix} + \begin{pmatrix} B \\ 0 \end{pmatrix} v(t), \tag{7}$$

$$y(t) = (C \ 0) \begin{pmatrix} x(t) \\ \varepsilon(t) \end{pmatrix},$$

where $\varepsilon(t)$ is the error of the estimator $\xi(t)$.

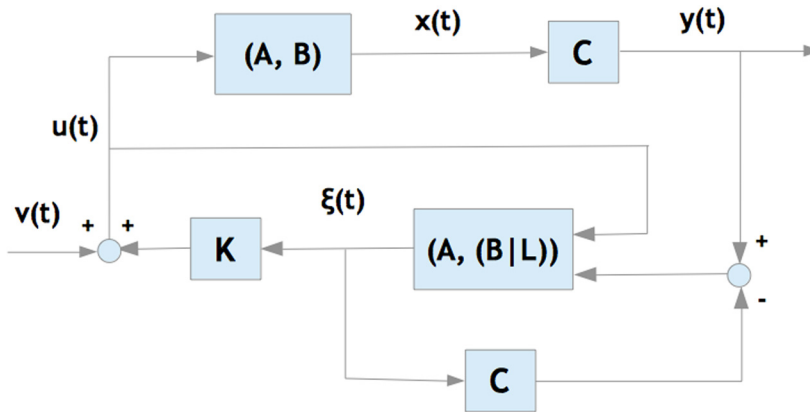


Figure 2: The dynamic compensator $(K, L) \in R^{m \times n} \times R^{n \times p}$ on linear system $(A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$.

Remark 2.2. It is worth highlighting that dynamic compensator $(K, L) \in R^{m \times n} \times R^{n \times p}$ brings linear system $(A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$ to linear system

$$\left(\begin{pmatrix} A + BK & -BK \\ 0 & A - LC \end{pmatrix}, \begin{pmatrix} B \\ 0 \end{pmatrix}, (C \ 0) \right) \in R^{2n \times 2n} \times R^{2n \times m} \times R^{p \times 2n}, \quad (8)$$

and therefore the characteristic polynomial of state-transition matrix of dynamic compensator is

$$\det(z\mathbb{1}_n - (A + BK)) \cdot \det(z\mathbb{1}_n - (A - LC)). \quad (9)$$

Hence, if we can assign the characteristic polynomials $\det(z\mathbb{1}_n - (A + BK))$ and $\det(z\mathbb{1}_n - (A - LC))$ from the triple (A, B, C) , then we are able to control effectively errors $\varepsilon(t)$, as well as states $x(t)$ by means of a dynamic compensator. This motivates the following definition:

Definition 2.3. Consider the linear system given by $\Sigma = (A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$. Then:

1. System Σ is pole-assignable if for each selection of $\lambda_1, \dots, \lambda_n \in R$ there exists some matrix $K \in R^{m \times n}$ such that $\det(z\mathbb{1}_n - (A + BK)) = (z - \lambda_1) \cdots (z - \lambda_n)$; that is to say, we are able to set the poles of the state transition matrix up to feedback action.
2. System Σ is state-estimable if for each selection of $\mu_1, \dots, \mu_n \in R$ there exists some matrix $L \in R^{n \times p}$ such that $\det(z\mathbb{1}_n - (A - LC)) = (z - \mu_1) \cdots (z - \mu_n)$; that is to say, we are able to set the poles of the state transition matrix up to output rejection.

Remark 2.4. Now it is clear from the aforementioned definition and the aforementioned construction that a dynamic compensator can be built from linear system $\Sigma = (A, B, C)$ if and only if Σ is both pole-assignable and state-estimable.

3 Reachability, controllability, and observability

Notions of reachability and observability are reviewed in this section. These conditions on a system (A, B, C) will be shown to be necessary for pole-assignability and state-estimability, and hence to build dynamic compensators.

Definition 3.1. Let R be a commutative ring and consider the triple of matrices $\Sigma = (A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$ representing a linear system given by equations (2). Then:

1. System Σ is reachable if and only if reachability block matrix $(B, AB, \dots, A^{n-1}B)$ defines an onto linear map

$$(R^m)^{\oplus n} \longrightarrow R^n.$$

2. System Σ is observable if and only if observability block matrix $\begin{pmatrix} C \\ CA \\ \vdots \\ CA^{n-1} \end{pmatrix}$ defines an injective linear map

$$R^n \longrightarrow (R^p)^{\oplus n}.$$

The following classical result characterizes when a linear system admits a dynamic compensator in terms of surjectivity of reachability map and injectivity of observability map.

Theorem 3.2. Let R be a commutative ring and $\Sigma = (A, B, C)$ a linear system, then:

1. If system Σ is pole assignable, then Σ is reachable.
2. If system Σ is state estimable, then Σ is observable.

If R is a field, then the reverse implications of the aforementioned statements also hold.

Proof. See [1, Theorem 3.1] for a proof of statement 1.

In order to prove (2) let $\Sigma = (A, B, C)$ be a state estimable linear system, then it is straightforward that system $\Sigma^t = (A^t, C^t, B^t)$ is pole assignable and hence (A^t, C^t) is a reachable pair by statement (1). Finally, note that if (A^t, C^t) is a reachable pair, then (A, C) is an observable pair and the result is complete. \square

Remark 3.3. System Σ^t is called the dual system. The aforementioned proof is an example of duality argument. Section 4 deals with dual systems and duality property.

Definition 3.4. [17, Def. 3.3.] A linear system is called canonical if it is reachable and observable.

It is clear that every canonical system over a field \mathbb{K} admits dynamic compensator. When does this result generalize to commutative rings? That is to say, we are interested in describing conditions for a commutative ring R in order that any canonical system over R admits a dynamic compensator.

Definition 3.5. A commutative ring R has the regulator property for canonical systems (or R is an RP-ring) if every canonical linear system admits a dynamic compensator.

Since every field has the regulator property it follows that fields are RP-rings.

Remark 3.6. In general, it is clear that a ring R having the regulator property verifies:

1. Every reachable system is pole-assignable, that is to say, R is a PA-ring.
2. Every observable system is state-estimable. In this case, we say that R is an SE-ring.

Note that the aforementioned classes of commutative rings are closely related. In fact, the class of PA-rings contains the class of SE-rings. In other words, we have the following characterization.

Theorem 3.7. *Let R be a commutative ring. Then the following are equivalent:*

1. R is an RP-ring.
2. R is a PA-ring and an SE-ring.
3. R is an SE-ring.

Proof. The only non-trivial issue to prove is that every SE-ring is a PA-ring. Suppose that R is an SE-ring and consider a reachable system $(A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$ and scalars $\lambda_1, \dots, \lambda_n \in R$. The result is proven if we find out a feedback matrix $K \in R^{m \times n}$ such that $\det(z\mathbb{1}_n - (A + BK)) = (z - \lambda_1) \cdots (z - \lambda_n)$.

Since (A, B, C) is reachable it follows that $(B, AB, \dots, A^{n-1}B)$ represents an onto linear map. Thus, the ideal $\mathcal{U}_n(B, AB, \dots, A^{n-1}B)$ generated by maximal minors (of size n) contains a unit of R . But

$$\mathcal{U}_n(B, AB, \dots, A^{n-1}B) = \mathcal{U}_n((B, AB, \dots, A^{n-1}B)^t) = \mathcal{U}_n \begin{pmatrix} B^t \\ B^t A^t \\ \vdots \\ B^t (A^t)^{n-1} \end{pmatrix}.$$

Thus, $\mathcal{U}_n \begin{pmatrix} B^t \\ B^t A^t \\ \vdots \\ B^t (A^t)^{n-1} \end{pmatrix}$ contains a unit of R and therefore $\begin{pmatrix} B^t \\ B^t A^t \\ \vdots \\ B^t (A^t)^{n-1} \end{pmatrix}$ represents an injective linear map and

consequently system (A^t, C^t, B^t) is observable. Now, since R is SE-ring, there exists $L \in R^{n \times m}$ such that $\det(z\mathbb{1}_n - (A^t + LB^t)) = (z - \lambda_1) \cdots (z - \lambda_n)$ and we are done by setting $K = L^t$. \square

4 Duality

The proofs of aforementioned results of Theorems 3.2, and 3.7, were performed by arguments of duality in linear systems [15,16]. Now we are dealing with main definitions and results on duality which are related to our problem.

Definition 4.1. Let $\Sigma = (A, B, C) \in R^{n \times n} \times R^{n \times m} \times R^{p \times n}$ be a linear system over commutative ring R representing the following right-hand-side equation together with a read-out equation

$$\begin{aligned} x(t+1) &= Ax(t) + Bu(t), \\ y(t) &= Cx(t). \end{aligned} \tag{10}$$

We denote by $\Sigma^t = (A^t, C^t, B^t) \in R^{n \times n} \times R^{n \times p} \times R^{m \times n}$ the *dual system* representing the right-hand-side equation together with a read-out equation

$$\begin{aligned} x^*(t+1) &= A^t x^*(t) + C^t y^*(t), \\ u^*(t) &= B^t x^*(t). \end{aligned} \tag{11}$$

Remark 4.2. Systems Σ and Σ^t are dual to each other. Note that it is straightforward to prove that there exists a dynamic compensator for Σ if and only if there exists a dynamic compensator for Σ^t .

For our purposes, we note that if R is a field, then Σ^t is canonical if and only if Σ is canonical. In the general case of R being a commutative ring one has the following result:

Theorem 4.3. *If system Σ is reachable, then dual system Σ^t is observable.*

Proof. See proof of Theorem 3.7. □

Remark 4.4. The converse does not hold: consider the trivial system over \mathbb{Z} given by $\Sigma = ((0), (1), (2))$. It is clear that Σ is observable because $(2) : \mathbb{Z} \rightarrow \mathbb{Z}$ is injective. But on the other hand dual system Σ^t is not reachable because $(2) : \mathbb{Z} \rightarrow \mathbb{Z}$ is not surjective.

Definition 4.5. [16] Let R be a commutative ring. We say that the duality principle holds for R , or R is a DP-ring if the following statements are equivalent:

1. System Σ is observable.
2. System Σ^t is reachable.

Remark 4.6. The aforementioned definition could also be read as: dual system of a canonical system is also canonical in DP-rings.

Remark 4.7. Recall that the class of DP-rings is characterized in the following terms [16, Corollary 2.5]: a commutative ring R is a DP-ring if and only if every finitely generated ideal of R with zero annihilator contains a unit.

5 SISO systems

This section is devoted to solve the regulator property for SISO linear systems. Recall that a SISO linear system over R is given by a triple $\Sigma = (A, \underline{b}, \underline{c}^t) \in R^{n \times n} \times R^{n \times 1} \times R^{1 \times n}$.

Theorem 5.1. *Let R be a commutative ring. The following statements are equivalent:*

- (i) *The regulator problem is solvable for every canonical SISO linear system over R .*
- (ii) *R equals its own total ring of fractions (i.e., $R = \mathcal{T}(R)$).*

Proof. Suppose (i) and let $x \in R$ be a non-zerodivisor. Then the SISO one-dimensional system $\Sigma = ((0), (1), (x))$ is canonical and consequently the regulator problem is solvable for Σ . Therefore, regulator problem is also solvable for Σ^t (see Remark 4.2). Hence, dual system $\Sigma^t = ((0), (x), (1))$ is in particular pole-assignable and therefore reachable. This yields that x is in fact a unit of R . Thus, every non zero divisor of R is a unit and R is a total ring of fractions.

Assume (ii) and let $\Sigma = (A, \underline{b}, \underline{c}^t) \in R^{n \times n} \times R^{n \times 1} \times R^{1 \times n}$ be a canonical linear system. Since Σ is in particular observable it follows that $(n \times n)$ -matrix $\begin{pmatrix} \underline{c}^t \\ \underline{c}^t A \\ \vdots \\ \underline{c}^t A^{n-1} \end{pmatrix}$ is injective. By McCoy's theorem, see [2, Theorem 4],

$\det \begin{pmatrix} \underline{c}^t \\ \underline{c}^t A \\ \vdots \\ \underline{c}^t A^{n-1} \end{pmatrix}$ is a nonzero divisor and consequently, by hypothesis, a unit. Therefore, the dual system

$\Sigma^t = (A^t, \underline{c}, \underline{b}^t)$ is also a canonical system.

Both systems $(A, \underline{b}, \underline{c}^t)$ and $(A^t, \underline{c}, \underline{b}^t)$ are reachable. But every reachable single-input system is pole assignable, see [1, Theorem 3.2]. Thus, both systems Σ and Σ^t are pole assignable and *a fortiori* the regulator problem is solvable for Σ . Then we conclude that regulator problem is solvable for every canonical SISO system. \square

6 RP-rings, SE-rings

The class of RP-rings has been characterized in Theorem 3.7 as the class of SE-rings. This class of rings contains the class of PA-rings. In this section, we will prove that SE-rings (or equivalently RP-rings) also contains DP-rings. Moreover, the class of RP-rings is exactly the intersection of PA-rings and DP-rings.

Theorem 6.1. *If R is an SE-ring, then R is a DP-ring. That is to say, if every observable system is state-estimable, then duality principle holds.*

Proof. Let $\Sigma = (A, B, C)$ be a observable linear system. Put $\lambda_1, \dots, \lambda_n \in R$. Since R is SE-ring it follows that there exists an output rejection L such that $\det(z\mathbb{1}_n - (A + LC)) = (z - \lambda_1) \cdots (z - \lambda_n)$. Hence, $\det(z\mathbb{1}_n - (A^t + C^t K)) = (z - \lambda_1) \cdots (z - \lambda_n)$ by setting $K = L^t$ and system (A^t, C^t, B^t) is pole-assignable. Then it follows that dual system $\Sigma^t = (A^t, C^t, B^t)$ is reachable and R is a DP-ring. \square

Theorem 6.2. *Let R be a commutative ring. Then the following are equivalent:*

1. *R is a RP-ring (or equivalently an SE-ring).*
2. *R is a PA-ring and a DP-ring.*

Proof. (1) \Rightarrow (2) is already proven in Theorems 3.7 and 6.1.

In order to prove the converse (2) \Rightarrow (1), consider a ring R which is a PA-ring and a DP-ring and let us prove that R is an SE-ring: Let $\Sigma = (A, B, C)$ be an observable system, consider scalars $\lambda_1, \dots, \lambda_n \in R$, and let us prove that there exists an output rejection L such that $\det(z\mathbb{1}_n - (A + LC)) = (z - \lambda_1) \cdots (z - \lambda_n)$:

Since R is DP it follows that $\Sigma^t = (A^t, C^t, B^t)$ is reachable and hence pole assignable because R is PA. Consider feedback K such that $\det(z\mathbb{1}_n - (A^t + C^t K)) = (z - \lambda_1) \cdots (z - \lambda_n)$; thus by setting $L = K^t$ one has $\det(z\mathbb{1}_n - (A + LC)) = (z - \lambda_1) \cdots (z - \lambda_n)$; therefore, there exists an output rejection, and consequently R is an SE-ring. \square

7 Some examples

The class of commutative rings where there exist dynamic compensators for canonical systems (RP-rings) is thus characterized as the class SE of rings where there exist output rejections for observable systems. Moreover, it is proven that the class SE is the intersection of classical classes of PA (pole-assignable) rings and DP (duality) rings.

Fields have been proved to be RP-rings, but we have not given any other example of a commutative ring in the class RP. In fact, Remark 4.4 proves that \mathbb{Z} is not an RP-ring by the lack of duality property.

Next result shows that any modular integer ring $\mathbb{Z}/d\mathbb{Z}$, and in general any finite ring, is an RP-ring and thus regulator problem is solvable on any finite ring. Thus, any canonical linear system with coefficients in a finite ring admits a dynamic compensator. This would be of interest in the field of codification theory, which actually often works with finite rings like $\mathbb{Z}/4\mathbb{Z}$; or more generally $\mathbb{Z}/p^r\mathbb{Z}$, where p is a prime integer. Reader can review codes over such Galois rings in [20, Ch. 7–8], and review decoding techniques as system tracking problems in [21].

Theorem 7.1. *Every finite ring is an RP-ring.*

Proof. Let R be a finite ring. Then by Remark 4.7 (or [16, Corollary 2.5]) it follows that R is DP-ring.

On the other hand, R is a (finite) product of (finite) local rings [20, Theorem 3.1.4]. Since local rings are PA-rings [1], it follows that R is a PA-ring because it is a finite product of PA-rings.

Therefore, R is an RP-ring because of R is both a DP-ring and a PA-ring. \square

The rest of the section is devoted to provide several negative examples in order to circumscribe classes of PA and DP rings.

Example 7.2. (Negative examples by lack of duality property) First note that finding out an example of PA ring that is not DP is quite easy because several wide classes of PA rings are known (apart from fields): local rings, one-dimensional \mathbb{K} -algebras, principal ideal rings, and so on. In order to assure that R is not DP-ring it suffices to find out an element $r \in R$ which is neither a unit nor a zerodivisor. Therefore, we obtain several examples of PA-rings that are not DP-rings like the ring \mathbb{Z} of integers (again), the ring of polynomials $\mathbb{K}[x]$, and the ring of formal power series $\mathbb{K}[[x]]$. All these rings fail to solve the regulator problem by means of lack of duality property.

Far more complicated is to find out an example of commutative ring which is a DP-ring but is not a PA-ring and hence fail to solve regulator problem by lack of pole-assignment.

Next, we expose a method for constructing reduced rings that verify the principle of duality but do not have PA property. This construction follows Huckaba's construction of $A + B$ rings in [22, §26].

Example 7.3. (A negative example by lack of pole-assignability) Let D be a reduced ring (no nonzero nilpotents) and \mathcal{P} a nonempty set of prime ideals of D . For any $\mathfrak{p} \in \mathcal{P}$ consider the domain D/\mathfrak{p} . Finally, define the commutative rings

$$B = \bigoplus_{n \in \mathbb{N}} \bigoplus_{\mathfrak{p} \in \mathcal{P}} D/\mathfrak{p}, \text{ and } V = \prod_{n \in \mathbb{N}} \prod_{\mathfrak{p} \in \mathcal{P}} D/\mathfrak{p}.$$

Consider the natural morphism $\varphi : D \rightarrow V$ sending $\varphi(d) = (d + \mathfrak{p})_{n,\mathfrak{p}}$, and $A = \text{Im}(\varphi : D \rightarrow V)$. Then $R = A + B$ is the Huckaba's construction based on D and \mathcal{P} .

The interesting properties of $A + B$ construction for our purposes are the following: If $\mathcal{P} = \text{Max}(D)$ is the set of maximal ideals of D , then:

- $R/B \cong D$.
- $R = \mathcal{T}(R)$ is a total ring of fractions.

Note that the selection of $\mathcal{P} \subseteq \text{Max}(D)$ yields that all factors of B and of V are in fact fields.

Theorem 7.4. *Let D be a reduced ring and $\mathcal{P} = \text{Max}(D)$ the set of maximal ideals of D . Then*

- (i) *The duality principle holds for Huckaba's construction $R = A + B$.*
- (ii) *If D is not a PA-ring, then R is not a PA-ring.*

Proof.

- (i) By [22, Theorems 26.4 and 27.1] $R = A + B$ equals its own total ring of fractions and every finitely generated ideal consisting entirely of zero divisors has a nonzero annihilator. Hence, R is a DP-ring.
- (ii) Since $D \cong R/B$ is not a PA-ring it follows that R is not a PA ring. This is clear because the residue class ring of a PA-ring is again a PA-ring, see [6, Theorem 1]. \square

Corollary 7.5. *Let D be a reduced ring, $\mathcal{P} = \text{Max}(D)$, and D is not PA-ring. Then Huckaba's construction $R = A + B$ is a DP-ring, but it is not a PA-ring (and a fortiori it fails to be RP only by lack of pole-assignment property).*

8 Conclusion

The dynamic compensator for a linear system over a commutative ring R is presented. Reachability and observability of the base triple are necessary conditions to set up a dynamic compensator (i.e., to solve the regulator problem). In the case of SISO systems, the class of rings where every canonical (reachable and observable) triple admits a regulator is exactly the class of total rings of fractions.

A commutative ring solves the regulator problem (i.e., is an RP-ring) if and only if it solves the output-rejection for observable systems (i.e., it is an SE-ring). The classical notion of PA-ring is weaker than dual notion of SE-ring. In fact, SE-ring = RP-ring = PA-ring + DP-ring. And there are examples of PA-rings which are not DP, and examples of DP-rings which are not PA.

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