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HOMOLOGY OF PRECROSSED MODULES

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ABSTRACT. We prove that the category of precrossed modules is an algebraic category, and we develop a cotriple (co)homology theory for precrossed modules which generalizes the Eilenberg-MacLane theory of (co)homology groups. We study the relationship of this theory with the (co)homology of crossed modules.

1. Introduction

From a certain point of view precrossed modules are generalizations of groups and they form a model of homotopy type in dimensions 1 and 2 for connected CW-complexes. The precrossed modules of nilpotence class 2 play an essential role in Baues' algebraic models of homotopy 3-types [3].

Conduché and Ellis [7] defined the first and second homology group of a precrossed P-module, and Inassaridze and Khmaladze [11] extended this theory by defining all homology groups modulo an integer q for a precrossed P-module in terms of nonabelian derived functors. Here P is a fixed action group.

In this paper we develop the basic elements to define cotriple homology and cohomology theories in the whole category of precrossed modules.

We start with the observation that the category of precrossed modules is an algebraic category; that is, there exists a tripleable forgetful functor from this category to the category of sets. The (co)homology theory for precrossed modules is therefore a particularization of the general theory of Barr and Beck [1].

We study this cotriple (co)homology theory and, in particular, the relationship between this theory and other classical or new (co)homology theories. We will show that this theory generalizes the Eilenberg-MacLane theory of (co)homology groups if we regard a group G as a precrossed module $(1, G, i)$

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or $(G,1,1)$. We also show that, in the particular case of a crossed module (T,G,∂), we obtain the low dimensional homology for crossed modules of Gilbert [9], and their generalizations in every dimension. Finally, we relate the theory to the CCG-(co)homology of crossed modules of Carrasco, Cegarra and R.-Grandjeán [5] through a long exact sequence connecting both theories.

We begin in Section 2 by recalling some standard results on the category of precrossed modules. Next, we show that this category is algebraic, and we generalize a result from [5] about tripleability for crossed modules. This leads to the notions of free and projective precrossed modules.

In Section 3 we define the cotriple (co)homology theory for precrossed modules, and we establish its basic properties.

Finally, in Section 4 we consider relationships with the other (co)homology theories described above.

2. Precrossed modules are an algebraic category

A *precrossed module* (M,P,μ) is a group homomorphism $\mu\colon M \longrightarrow P$ together with an action of P on M, denoted by ${}^pm$ for $p \in P$ and $m \in M$, which satisfies $\mu({}^pm) = p\mu(m)p^{-1}$ for all $p \in P$ and $m \in M$. If, in addition, μ verifies Peiffer's identity ${}^{\mu(m)}m' = mm'm^{-1}$ for all $m, m' \in M$, (M,P,μ) is a said to be a *crossed module*.

Examples.

(i) Let N be a normal subgroup of G. The inclusion homomorphism $i\colon N \longrightarrow G$ with the action ${}^gn = gng^{-1}$, $g \in G$, $n \in N$, is a crossed module. In particular, (G,G,Id) and $(1,G,i)$ are crossed modules.

(ii) If A is an ordinary ZG-module, then $(A,G,0)$ is a crossed module.

(iii) Let P, G be groups with P acting on G non-trivially, or G a non-abelian group. Then $(G \rtimes P, P, \pi)$ is a precrossed module, but not a crossed module, where $G \rtimes P$ is the semidirect product of G and P, π is the natural surjective homomorphism of $G \rtimes P$ on P and the action of P on $G \rtimes P$ is ${}^{p'}(g,p) = ({}^{p'}g, {}^{p'}p)$. In particular, if G is a non-abelian group, $(G,1,1)$ is a precrossed module, but not a crossed module.

A *precrossed module morphism* $(\Phi,\Psi)\colon (M_1,P_1,\mu_1) \longrightarrow (M_2,P_2,\mu_2)$ is a pair of group homomorphisms $\Phi\colon M_1 \longrightarrow M_2$ and $\Psi\colon P_1 \longrightarrow P_2$ such that $\Psi \circ \mu_1 = \mu_2 \circ \Phi$ and $\Phi({}^pm) = {}^{\Psi(p)}\Phi(m)$ for all $p \in P_1$ and $m \in M_1$.

We denote the category of precrossed modules by $\mathcal{PCM}$.

A morphism (Φ,Ψ) in $\mathcal{PCM}$ is said to be *injective* (resp. *surjective*) if both Φ and Ψ are injective (resp. surjective) group homomorphisms.

A *precrossed submodule* (N,Q,μ') of a precrossed module (M,P,μ) is a precrossed module such that N and Q are subgroups of M and P, respectively, the action of Q on N is induced by that of P on M, and $\mu_{|N} = \mu'$. A submodule is said to be a *normal precrossed submodule* if, in addition, N and

Q are normal subgroups in M and P, respectively, and if, for all $p \in P$, $q \in Q$, $m \in M$ and $n \in N$, ${}^{p}n \in N$ and ${}^{q}mm^{-1} \in N$.

If (N, Q, μ) is a normal precrossed submodule of (M, P, μ), we define the *quotient precrossed module* $(M, P, \mu)/(N, Q, \mu)$ as $(M/N, P/Q, \overline{\mu})$, where the homomorphism $\overline{\mu}$ is induced by μ and P/Q acts on M/N by ${}^{pQ}mN = ({}^{p}m)N$ for $p \in P$ and $m \in M$.

We call the *Peiffer subgroup* $\langle M, M\rangle$ of a precrossed module (M, P, μ) the subgroup of M generated by the Peiffer elements $m_1 m_2 m_1^{-1}\ {}^{\mu(m_1)}m_2^{-1}$, $m_1, m_2 \in M$. The Peiffer subgroup is a normal subgroup of M, and the quotient precrossed module $(M, P, \mu)/(\langle M, M\rangle, 1, 1)$ is a crossed module called the *Peiffer abelianisation* of (M, P, μ).

The *kernel* of a precrossed module morphism $(\Phi, \Psi)\colon (M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2)$ is the normal precrossed submodule $(\operatorname{Ker}\Phi, \operatorname{Ker}\Psi, \mu_1)$ of (M_1, P_1, μ_1). Its *image* is the precrossed submodule $(\operatorname{Im}\Phi, \operatorname{Im}\Psi, \mu_2)$ of (M_2, P_2, μ_2).

If two precrossed modules (M_1, P_1, μ_1) and (M_2, P_2, μ_2) are given, their *product* is the precrossed module $(M_1 \times M_2, P_1 \times P_2, \mu_1 \times \mu_2)$, with $P_1 \times P_2$ acting on $M_1 \times M_2$ by ${}^{(p_1,p_2)}(m_1, m_2) = ({}^{p_1}m_1, {}^{p_2}m_2)$ for $p_1 \in P_1$, $p_2 \in P_2$, $m_1 \in M_1$, and $m_2 \in M_2$.

We next introduce analogues of some basic concepts from group theory, such as the centre or commutator groups, in the category of precrossed modules.

We call the *centre* $Z(M, P, \mu)$ of a precrossed module (M, P, μ) the normal precrossed submodule $(\operatorname{Inv}(M) \cap Z(M), \operatorname{St}_P(M) \cap Z(P), \mu)$, where $\operatorname{St}_P(M)$ denotes the group $\{p \in P \mid {}^{p}m = m \text{ for all } m \in M\}$, $\operatorname{Inv}(M) = \{m \in M \mid \mu(m) \in \operatorname{St}_P(M) \text{ and } {}^{p}m = m \text{ for all } p \in P\}$ and $Z(M)$ and $Z(P)$ denote the centres of M and P, respectively. Using the same argument as in [13], one can show that $Z(M, P, \mu)$ is the maximal central precrossed submodule of (M, P, μ). Hence $Z(M, P, \mu)$ coincides with the categorical notion of centre developed by Huq [10].

A precrossed module (M, P, μ) is said to be *abelian* if it satisfies $(M, P, \mu) = Z(M, P, \mu)$, or, equivalently, if M and P are abelian groups and P acts trivially on M. The variety of abelian precrossed modules $\mathcal{APCM}$ coincides with that of abelian crossed modules. We can identify it with the category of homomorphisms of abelian groups, which is equivalent to the category of right modules over the ring of matrices $\begin{pmatrix} \mathbb{Z} & 0 \\ \mathbb{Z} & \mathbb{Z} \end{pmatrix}$ (see [5]).

If (N, Q, μ) is a normal precrossed submodule of (M, P, μ), we define the *commutator precrossed submodule* $[(N, Q, \mu), (M, P, \mu)]$ of (N, Q, μ) and (M, P, μ) as the normal precrossed submodule $([N, M][Q, M][P, N], [P, Q], \mu)$ of (M, P, μ), where $[P, N]$ denotes the normal subgroup of M generated by the elements $\{{}^{p}nn^{-1} \mid p \in P, n \in N\}$, $[Q, M]$ denotes the normal subgroup of M generated by the elements $\{{}^{q}mm^{-1} \mid q \in Q, m \in M\}$, and $[N, M]$ and

$[P,Q]$ denote the usual commutator subgroups of N with M and P with Q, respectively.

In particular, the *commutator precrossed submodule* $[(M,P,\mu),(M,P,\mu)]$ of a precrossed module (M,P,μ) is the normal precrossed submodule $([M,M][P,M],[P,P],\mu)$ of (M,P,μ). It is the smallest precrossed module of (M,P,μ) for which the quotient is an abelian precrossed module.

To prove the tripleability of $\mathcal{PCM}$ over $\mathcal{S}et$ we will use the following criterion, due to Linton [8]; for background on tripleability we refer to [2].

THEOREM 2.1. *A functor $U\colon D \longrightarrow \mathcal{S}et$ is tripleable if and only if U has a left adjoint and the following three conditions are satisfied:*

(a) *D has kernel pairs and coequalizers.*
(b) *$p\colon Y \longrightarrow Z$ is a coequalizer $\Longleftrightarrow$ $Up\colon UY \longrightarrow UZ$ is a coequalizer.*
(c) *$X \underset{t}{\overset{s}{\rightrightarrows}} Y$ is a kernel pair $\Longleftrightarrow$ $UX \underset{Ut}{\overset{Us}{\rightrightarrows}} UY$ is a kernel pair, where p, s and t denote morphisms in D.*

We will apply this criterion to the forgetful functor $\mathcal{U}\colon \mathcal{PCM} \longrightarrow \mathcal{S}et$, $\mathcal{U}(M,P,\mu) = M \times P$, which assigns to each precrossed module (M,P,μ) the cartesian product of the underlying sets M and P.

PROPOSITION 2.2. *The functor $\mathcal{U}\colon \mathcal{PCM} \longrightarrow \mathcal{S}et$ has a left adjoint.*

Proof. It is known (see [12]) that the forgetful functor

$$U_1\colon \mathcal{PCM} \longrightarrow \mathcal{G}rp \downarrow \mathcal{G}rp,\ U_1(M,P,\mu) = \mu$$

has a left adjoint by the functor

$$F_1\colon \mathcal{G}rp \downarrow \mathcal{G}rp \longrightarrow \mathcal{PCM}, \quad F_1(G \xrightarrow{\lambda} H) = (\overline{G}, H, \langle \lambda, \mathrm{Id}\rangle_{|\overline{G}}),$$

where

$$\overline{G} = \mathrm{Ker}(G * H \xrightarrow{\langle 0,\mathrm{Id}\rangle} H), \quad \langle \lambda, \mathrm{Id}\rangle\colon G * H \longrightarrow H,$$

and H acts on $\overline{G}$ by conjugation. On the other hand, the functor

$$U_2\colon \mathcal{G}rp \downarrow \mathcal{G}rp \longrightarrow \mathcal{G}rp,\ U_2(G \xrightarrow{\lambda} H) = G \times H$$

has a left adjoint by the functor $F_2\colon \mathcal{G}rp \longrightarrow \mathcal{G}rp \downarrow \mathcal{G}rp$, which takes a group G into $F_2(G) = (G \xrightarrow{i_1} G * G)$, the first inclusion in the coproduct.

We denote by $U_3\colon \mathcal{G}rp \longrightarrow \mathcal{S}et$ the usual forgetful functor and by $F_3\colon \mathcal{S}et \longrightarrow \mathcal{G}rp$ its left adjoint, the free group functor.

Composing these three adjunctions,

$$\mathcal{S}et \underset{U_3}{\overset{F_3}{\rightleftarrows}} \mathcal{G}rp \underset{U_2}{\overset{F_2}{\rightleftarrows}} \mathcal{G}rp \downarrow \mathcal{G}rp \underset{U_1}{\overset{F_1}{\rightleftarrows}} \mathcal{PCM},$$

we see that $\mathcal{U} = U_3 \circ U_2 \circ U_1 \colon \mathcal{PCM} \longrightarrow \mathcal{S}et$ is right adjoint to the functor

$$\mathcal{F} = F_1 \circ F_2 \circ F_3\colon \mathcal{S}et \longrightarrow \mathcal{PCM}, \quad \mathcal{F}(X) = (\overline{F}, F * F, \langle i_1, \mathrm{Id}\rangle_{|\overline{F}}),$$

where $F = F_3(X)$ is the free group over X,

$$\overline{F} = \operatorname{Ker}(F * (F * F) \overset{\langle 0,\mathrm{Id}\rangle}{\longrightarrow} F * F), \quad \langle i_1, \mathrm{Id}\rangle \colon F * (F * F) \longrightarrow F * F,$$

and $F * F$ acts on $\overline{F}$ by conjugation. □

THEOREM 2.3. *The functor* $\mathcal{U} \colon \mathcal{PCM} \longrightarrow \mathcal{S}et$ *is tripleable.*

Proof. We check that the conditions of Theorem 2.1 are satisfied:

(a) $\mathcal{PCM}$ *has kernel pairs.* It is easy to check that the kernel pair of a precrossed module morphism $(M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2)$ is the precrossed submodule $(M_1 \underset{M_2}{\times} M_1, P_1 \underset{P_2}{\times} P_1, \mu_1 \times \mu_1)$ of the product $(M_1, P_1, \mu_1) \times (M_1, P_1, \mu_1)$, with the projections to the first and second components given by

$$(M_1 \underset{M_2}{\times} M_1, P_1 \underset{P_2}{\times} P_1, \mu_1 \times \mu_1) \underset{(\pi_M^2, \pi_P^2)}{\overset{(\pi_M^1, \pi_P^1)}{\rightrightarrows}} (M_1, P_1, \mu_1).$$

$\mathcal{PCM}$ *has coequalizers.* Given a pair of morphisms

$$(M_1, P_1, \mu_1) \underset{(\Psi_M, \Psi_P)}{\overset{(\Phi_M, \Phi_P)}{\rightrightarrows}} (M_2, P_2, \mu_2),$$

consider the least normal precrossed submodule (N, Q, μ_2) of (M_2, P_2, μ_2) such that the canonical projection

$$(\pi_N, \pi_Q) \colon (M_2, P_2, \mu_2) \longrightarrow (M_2, P_2, \mu_2) \diagup (N, Q, \mu_2)$$

satisfies $(\pi_N, \pi_Q) \circ (\Phi_M, \Phi_P) = (\pi_N, \pi_Q) \circ (\Psi_M, \Psi_P)$. Such a submodule exists since the intersection (defined in the obvious way) of normal precrossed submodules satisfying the above condition is a normal precrossed submodule that also satisfies this condition. It is also clear that (π_N, π_Q) is the coequalizer of (Φ_M, Φ_P) and (Ψ_M, Ψ_P).

(b) $\mathcal{U}$ preserves and reflects coequalizers because both in $\mathcal{PCM}$ and in $\mathcal{S}et$ the coequalizers are surjective morphisms (a surjective precrossed module morphism is the coequalizer of its kernel inclusion and the zero map).

(c) $\mathcal{U}$ preserves kernel pairs because it has a left adjoint. To see that $\mathcal{U}$ reflects kernel pairs, first note that by a *congruence* on a precrossed module (M, P, μ) we mean a precrossed submodule $(R, C, \mu \times \mu)$ of the product precrossed module $(M, P, \mu) \times (M, P, \mu) = (M \times M, P \times P, \mu \times \mu)$ such that both $R \subset M \times M$ and $C \subset P \times P$ are equivalence relations. As in groups, giving a congruence $(R, C, \mu \times \mu)$ on a precrossed module (M, P, μ) is equivalent to giving a normal precrossed submodule $(N_R, Q_C, \mu) \lhd (M, P, \mu)$, where $N_R = \{m \in M \mid (1, m) \in R\}$ and $Q_C = \{p \in P \mid (1, p) \in C\}$. In fact,

$$(R, C, \mu \times \mu) \underset{(\pi_M^2, \pi_P^2)}{\overset{(\pi_M^1, \pi_P^1)}{\rightrightarrows}} (M, P, \mu)$$

is the kernel pair of the quotient

$$(M, P, \mu) \longrightarrow (M, P, \mu) \diagup (N_R, Q_C, \mu).$$

Take a pair

$$(M_1, P_1, \mu_1) \underset{(\Psi_M, \Psi_P)}{\overset{(\Phi_M, \Phi_P)}{\rightrightarrows}} (M_2, P_2, \mu_2)$$

of precrossed module morphisms which is a kernel pair in $\mathcal{S}et$. This implies that

$$(\Phi_M \times \Phi_P) \times (\Psi_M \times \Psi_P) \colon M_1 \times P_1 \longrightarrow (M_2 \times P_2) \times (M_2 \times P_2)$$

is injective, and its image is an equivalence relation on the group $M_2 \times P_2$. Thus

$$(\Phi_M, \Phi_P) \times (\Psi_M, \Psi_P) \colon (M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2) \times (M_2, P_2, \mu_2)$$

is an injective morphism that establishes a precrossed module isomorphism between (M_1, P_1, μ_1) and its image, denoted by $(R, C, \mu_2 \times \mu_2)$. It is clear that $(R, C, \mu_2 \times \mu_2)$ is a congruence on (M_2, P_2, μ_2). We therefore have the following commutative diagram:

$$\begin{array}{ccc} (M_1, P_1, \mu_1) & \underset{(\Psi_M, \Psi_P)}{\overset{(\Phi_M, \Phi_P)}{\rightrightarrows}} & (M_2, P_2, \mu_2) \\ & \searrow & {\scriptstyle (\pi^1_M, \pi^1_P)} \upuparrows {\scriptstyle (\pi^2_M, \pi^2_P)} \\ & & (R, C, \mu_2 \times \mu_2) \end{array}$$

This yields the result. □

Observe that every Birkhoff subvariety of $\mathcal{PCM}$ (which is closed under subobjects, quotients and products) is also closed under the operations of taking kernel pairs and coequalizers, and by Theorem 2.1 is therefore also algebraic. As a corollary, we obtain the following result from [5]:

COROLLARY 2.4. *The category of crossed modules $\mathcal{CM}$ is tripleable over $\mathcal{S}et$, via the forgetful functor $\nu \colon \mathcal{CM} \longrightarrow \mathcal{S}et$, $\nu(T, G, \partial) = T \times G$.*

The existence of a good pair of adjoint functors between $\mathcal{PCM}$ and $\mathcal{S}et$ will allow us to establish some useful facts on free and projective precrossed modules.

Recall that an object P in a category is said to be *projective* if for every regular epimorphism (i.e., a coequalizer) $p \colon A \twoheadrightarrow B$, $p_* \colon \operatorname{Hom}(P, A) \longrightarrow \operatorname{Hom}(P, B)$ is a surjective map. A category is said to have *enough projective objects* if for each object Y there exists a regular epimorphism $P \twoheadrightarrow Y$ with P a projective object (that is, a *projective presentation* of Y).

As was shown in Theorem 2.3, regular epimorphisms in $\mathcal{PCM}$ are just surjective morphisms. Thus, for each set X, the free precrossed module (relative to U) $\mathcal{F}(X)$ is a projective precrossed module, and every precrossed module (M,P,μ) has a projective presentation through the counit of the adjunction $\mathcal{F}\mathcal{U}(M,P,\mu) \twoheadrightarrow (M,P,\mu)$.

There are more projective objects in $\mathcal{PCM}$. In the following proposition we give a family of projective objects with some members not isomorphic to any value of $\mathcal{F}$.

Proposition 2.5. *Let P and Q be free groups. Then the precrossed module $(\overline{P}, P*Q, \langle i_P, \mathrm{Id}\rangle_{|\overline{P}})$ is projective, where $\overline{P} = \mathrm{Ker}(P*(P*Q) \xrightarrow{\langle 0,\mathrm{Id}\rangle} P*Q)$, $\langle i_P, \mathrm{Id}\rangle\colon P*(P*Q) \longrightarrow P*Q$, and $P*Q$ acts on $\overline{P}$ by conjugation.*

Proof. The inclusion in the coproduct $i_P\colon P \rightarrowtail P*Q$ is a projective object in the category of group homomorphisms $\mathcal{G}rp \downarrow \mathcal{G}rp$. Take

$$\alpha = (\alpha_1, \alpha_2)\colon (P \xrightarrow{i_P} P*Q) \longrightarrow (G' \xrightarrow{\lambda'} H'),$$
$$\beta = (\beta_1, \beta_2)\colon (G \xrightarrow{\lambda} H) \longrightarrow (G' \xrightarrow{\lambda'} H'),$$

which are morphisms in $\mathcal{G}rp \downarrow \mathcal{G}rp$, with β_1 and β_2 surjective homomorphisms. Since P and Q are free groups, there exist $\varepsilon\colon P \longrightarrow G$ and $\phi''\colon Q \longrightarrow H$ such that $\beta_1 \circ \varepsilon = \alpha_1$ and $\beta_2 \circ \phi'' = \alpha_2 \circ i_Q$. Also, $\phi' = \lambda \circ \varepsilon$ and ϕ'' induce $\phi\colon P*Q \longrightarrow H$ such that $\beta_2 \circ \phi = \alpha_2$. Finally, the pair $(\varepsilon, \phi)\colon (P \xrightarrow{i_P} P*Q) \longrightarrow (G \xrightarrow{\lambda} H)$ is a morphism in $\mathcal{G}rp \downarrow \mathcal{G}rp$, because $\phi \circ i_P = \lambda \circ \varepsilon$, and we have $\alpha = \beta \circ (\varepsilon, \phi)$.

Since the forgetful functor $U_1\colon \mathcal{PCM} \longrightarrow \mathcal{G}rp \downarrow \mathcal{G}rp$ (see Theorem 2.3) preserves surjective morphisms, its left adjoint $F_1\colon \mathcal{G}rp \downarrow \mathcal{G}rp \longrightarrow \mathcal{PCM}$ preserves projective objects, and so $F_1(P \xrightarrow{i_P} P*Q)$ is a projective precrossed module. □

Note that free crossed modules relative to $\nu\colon \mathcal{CM} \longrightarrow \mathcal{S}et$ (see [5]) are obtained from free precrossed modules (relative to $\mathcal{U}\colon \mathcal{PCM} \longrightarrow \mathcal{S}et$), which yields its Peiffer abelianisation. Similarly, if we take the quotient of a projective precrossed module by its Peiffer subgroup, we get a projective crossed module.

A non-zero free precrossed module $(\overline{F}, F*F, \langle i_1, \mathrm{Id}\rangle_{|\overline{F}})$ is never a crossed module. In fact, any projective precrossed module in the family described in Proposition 2.5, i.e., $(\overline{P}, P*Q, \langle i_P, \mathrm{Id}\rangle_{|\overline{P}})$ with $P \neq 0$, is not a crossed module. To see this, take a non-trivial element $p \in P$; then it is clear that ${}^{\langle i_P, \mathrm{Id}\rangle(p)}p \neq ppp^{-1} = p$.

Observe that every projective precrossed module (M,P,μ) is a retract of a free precrossed module, and so M and P are free groups.

3. Cotriple (co)homology of precrossed modules

In this section we particularize to our context the cotriple (co)homology of Barr and Beck [1]. Before introducing homology and cohomology theories for precrossed modules, we study the category of abelian group objects in the category of precrossed modules.

The category of abelian group objects in a category plays a fundamental role in the description of the homology and the cohomology of the category. It was shown in [5] that the abelian group objects in the category $\mathcal{CM}$ are just the abelian crossed modules (i.e., those which coincide with its centre), as the abelian group objects in the category $\mathcal{G}rp$ are simply the abelian groups.

Proposition 3.1. *The forgetful functor $J\colon \mathcal{APCM} \longrightarrow \mathcal{PCM}$ is a full and faithful embedding whose replete image consists of abelian precrossed modules.*

Proof. The proof is analogous to that given in [5] for the case of crossed modules; it suffices to replace $\mathcal{CM}$ by $\mathcal{PCM}$. □

The inclusion J of the variety $\mathcal{APCM}$ in $\mathcal{PCM}$ has a left adjoint (the reflector) $ab\colon \mathcal{PCM} \longrightarrow \mathcal{APCM}$ called the *abelianisation functor*, which assigns to a precrossed module (M,P,μ) the abelian precrossed module $(M,P,\mu)_{ab} = (M/[M,M][P,M], P/[P,P], \overline{\mu})$.

The functors $\mathcal{F}\colon \mathcal{S}et \longrightarrow \mathcal{PCM}$ and $\mathcal{U}\colon \mathcal{PCM} \longrightarrow \mathcal{S}et$ described in Section 2 induce a free cotriple (C,δ,ε) in $\mathcal{PCM}$, with $C = \mathcal{F}\circ\mathcal{U}\colon \mathcal{PCM} \longrightarrow \mathcal{PCM}$, $\delta\colon C \Rightarrow \mathrm{Id}_{\mathcal{PCM}}$ the counit of the adjunction and $\varepsilon\colon C \Rightarrow C^2$ the comultiplication. Every precrossed module (M,P,μ) has a *standard free simplicial resolution* $C.(M,P,\mu) \longrightarrow (M,P,\mu)$, where $C.(M,P,\mu)$ is the simplicial precrossed module with the n-dimensional object $C_n(M,P,\mu) = C^{n+1}(M,P,\mu)$, for $n \geq 0$, and with face and degeneracy operators

$$d_i = C^{n-i}\delta C^i(M,P,\mu)\colon C_n(M,P,\mu) \longrightarrow C_{n-1}(M,P,\mu), 0 \leq i \leq n,$$

$$s_i = C^{n-i}\varepsilon C^i(M,P,\mu)\colon C_n(M,P,\mu) \longrightarrow C_{n+1}(M,P,\mu), 0 \leq i \leq n.$$

Applying the abelianisation functor $ab\colon \mathcal{PCM} \longrightarrow \mathcal{APCM}$, we obtain an augmented simplicial complex of abelian precrossed modules

$$(C.(M,P,\mu))_{ab} \twoheadrightarrow (M,P,\mu)_{ab}.$$

If we take the alternating sum of the abelianised face operators we get the chain complex of abelian precrossed modules

$$\dots (C_n(M,P,\mu))_{ab} \xrightarrow{\partial_n} (C_{n-1}(M,P,\mu))_{ab} \cdots \xrightarrow{\partial_1} (C(M,P,\mu))_{ab} \longrightarrow 0$$

whose homology provides the *homology groups* of the precrossed module (M,P,μ):

$$H_n(M,P,\mu) = H_{n-1}((C.(M,P,\mu))_{ab}, \partial_*), \quad n \geq 1.$$

On the other hand, if (A, B, f) is an abelian precrossed module, applying the functor $\mathrm{Hom}_{\mathcal{PCM}}(-, (A, B, f))$ to $C.(M, P, \mu)$, we obtain an augmented simplicial complex of abelian groups

$$\mathrm{Hom}_{\mathcal{PCM}}((M, P, \mu), (A, B, f)) \rightarrowtail \mathrm{Hom}_{\mathcal{PCM}}(C.(M, P, \mu), (A, B, f))$$

and the chain complex of abelian groups

$$\begin{aligned} 0 &\longrightarrow \mathrm{Hom}_{\mathcal{PCM}}(C(M, P, \mu), (A, B, f)) \xrightarrow{\partial^1} \cdots \\ \cdots &\longrightarrow \mathrm{Hom}_{\mathcal{PCM}}(C_n(M, P, \mu), (A, B, f)) \xrightarrow{\partial^n} \\ &\xrightarrow{\partial^n} \mathrm{Hom}_{\mathcal{PCM}}(C_{n+1}(M, P, \mu), (A, B, f)) \cdots \end{aligned}$$

We define the *cohomology groups of* (M, P, μ) *with coefficients in* (A, B, f) for $n \geq 1$ by

$$\begin{aligned} H^n((M, P, \mu), (A, B, f)) &= H^{n-1}(\mathrm{Hom}_{\mathcal{PCM}}(C.(M, P, \mu), (A, B, f)), \partial^*) \\ &= H^{n-1}(\mathrm{Hom}_{\mathcal{PCM}}((C.(M, P, \mu))_{ab}, (A, B, f)), \partial^*). \end{aligned}$$

Note that the homology groups of a precrossed module are actually abelian group objects in $\mathcal{PCM}$.

Proposition 3.2.

(i) $H_n(-)\colon \mathcal{PCM} \longrightarrow \mathcal{APCM}$ *and* $H^n(-,-)\colon \mathcal{PCM}^{op} \times \mathcal{APCM} \longrightarrow \mathcal{A}b$ *are functors for each* $n \geq 1$.

(ii) *For every precrossed module* (M, P, μ) *and all abelian precrossed modules* (A, B, f),

$$\begin{aligned} H_1(M, P, \mu) &= (M, P, \mu)_{ab}, \\ H^1((M, P, \mu), (A, B, f)) &= \mathrm{Hom}_{\mathcal{PCM}}((M, P, \mu), (A, B, f)) \\ &= \mathrm{Hom}_{\mathcal{PCM}}((M, P, \mu)_{ab}, (A, B, f)). \end{aligned}$$

(iii) *If* (M, P, μ) *is a projective precrossed module, then*

$$H_n(M, P, \mu) = 0 = H^n((M, P, \mu), (A, B, f))$$

for each abelian precrossed module (A, B, f) *and* $n \geq 2$.

(iv) *If* $0 \longrightarrow (A_1, B_1, f_1) \longrightarrow (A, B, f) \longrightarrow (A_2, B_2, f_2) \longrightarrow 0$ *is exact, then there exists a long exact sequence in cohomology:*

$$\begin{aligned} \ldots \longrightarrow H^n((M, P, \mu), (A_1, B_1, f_1)) &\longrightarrow H^n((M, P, \mu), (A, B, f)) \\ \longrightarrow H^n((M, P, \mu), (A_2, B_2, f_2)) &\longrightarrow H^{n+1}((M, P, \mu), (A_1, B_1, f_1)) \ldots \end{aligned}$$

Remark 1. An alternative way for computing the (co)homology consists of replacing the standard free simplicial resolution by a *projective simplicial resolution* (see [1]), that is, an augmented simplicial complex of precrossed modules $(M., P., \mu.) \longrightarrow (M, P, \mu)$, where (M_n, P_n, μ_n) is a projective precrossed module for each $n \geq 0$, and the simplicial set

$$\mathcal{U}(M., P., \mu.) = M. \times P. \longrightarrow \mathcal{U}(M, P, \mu) = M \times P$$

has a contraction.

It was shown in [1] that

$$H_n(M, P, \mu) = H_{n-1}((M., P., \mu.)_{ab}, \partial_*)$$

and

$$H^n((M, P, \mu), (A, B, f)) = H^{n-1}(\mathrm{Hom}_{\mathcal{PCM}}((M., P., \mu.), (A, B, f)), \partial^*)$$

for each abelian precrossed module (A, B, f) and all $n \geq 1$.

Observe that $\mathcal{U}(M., P., \mu.) \longrightarrow \mathcal{U}(M, P, \mu)$ is obviously a Kan complex, so $\mathcal{U}(M., P., \mu.) \longrightarrow \mathcal{U}(M, P, \mu)$ has a contraction if and only if $M. \longrightarrow M$ and $P. \longrightarrow P$ are both weak equivalences, i.e., if the homotopy groups verify $\pi_i(M.) = 0 = \pi_i(P.)$ for $i \geq 1$, and $\pi_0(M.) = M$, $\pi_0(P.) = P$.

4. Connection to other (co)homology theories

We begin with a theorem which shows the main connection between our (co)homology theory for precrossed modules and the Eilenberg-MacLane theory of (co)homology groups.

Denote by $H_n(G) = H_n(G, Z)$ the *nth integral homology group* of a group G, and by $H^n(G, A)$ the *nth cohomology group* of a group G with coefficients in an abelian group A (considered as a trivial G-module).

The category of groups may be regarded as a subcategory of the category of precrossed modules, with the inclusion given by the functor $i\colon \mathcal{G}rp \longrightarrow \mathcal{PCM}$, $i(G) = (1, G, i)$. The functor i has a right adjoint $k\colon \mathcal{PCM} \longrightarrow \mathcal{G}rp$, $k(M, P, \mu) = P$.

Another common way of thinking of a group as a precrossed module is through the functor $\varepsilon\colon \mathcal{G}rp \longrightarrow \mathcal{PCM}$, $\varepsilon(G) = (G, G, id)$, where G acts on itself by conjugation. The functor ε has as left adjoint the functor k.

THEOREM 4.1.

(i) *For every group G and every abelian precrossed module (A, B, f),*

$$H_n(iG) \cong iH_n(G),\ H^n(iG, (A, B, f)) \cong H^n(G, B),\ n \geq 1.$$

(ii) *For every precrossed module (M, P, μ) and all $n \geq 1$,*

$$kH_n(M, P, \mu) \cong H_n(P).$$

(iii) *For every precrossed module (M, P, μ), every abelian group A, and all $n \geq 1$,*

$$H^n((M, P, \mu), \varepsilon A) \cong H^n(P, A).$$

Proof. As was shown in [1], Eilenberg-MacLane (co)homology groups can be computed through free simplicial resolutions. That is, given a group G, if $F. \twoheadrightarrow G$ is an augmented simplicial group such that F_n is a free group for every $n \geq 0$, $\pi_i(F.) = 0$ for $i > 0$, and $\pi_0(F.) = G$, then

$$H_n(G) \cong H_{n-1}((F.)_{ab}, \partial_*),$$
$$H^n(G, A) \cong H^{n-1}(\mathrm{Hom}_{\mathcal{G}rp}((F.), A), \partial^*)$$

for every abelian group A and every $n \geq 1$, where the differential is taken to be the alternating sum of the abelianised face operators.

To prove (i) take a free simplicial resolution $F. \twoheadrightarrow G$ of the group G. By Proposition 2.5, the augmented simplicial precrossed module $iF. \twoheadrightarrow iG$ is a projective simplicial resolution of the precrossed module iG. Thus,

$$\begin{aligned} H_n(iG) &\cong H_{n-1}((iF.)_{ab}) \cong H_{n-1}(i(F._{ab})) \\ &\cong iH_{n-1}((F.)_{ab}) \cong iH_n(G),\ n \geq 1, \\ H^n(iG, (A, B, f)) &\cong H^{n-1}(\mathrm{Hom}_{\mathcal{PCM}}((iF.), (A, B, f))) \\ &\cong H^{n-1}(\mathrm{Hom}_{\mathcal{G}rp}(F., k(A, B, f))) \cong H^{n-1}(\mathrm{Hom}_{\mathcal{G}rp}(F., B)) \\ &\cong H^n(G, B),\ n \geq 1. \end{aligned}$$

Here we used the fact that i preserves kernels and cokernels and commutes with the abelianisation functors (i.e., $i \circ ab = ab \circ i$).

Now we prove (ii) and (iii). Recall that, as shown at the end of last section, for every precrossed module (M, P, μ) the simplicial group $k(C.(M, P, \mu))$ is a free simplicial resolution of the group P. Thus

$$\begin{aligned} kH_n(M, P, \mu) = kH_{n-1}(C.(M, P, \mu)_{ab}) &\cong H_{n-1}(k(C.(M, P, \mu))_{ab}) \\ &\cong H_n(P),\ n \geq 1, \end{aligned}$$

since $k \circ ab = ab \circ k$. Similarly, for every abelian group A we have

$$\begin{aligned} H^n((M, P, \mu), \varepsilon A) &= H^{n-1}(\mathrm{Hom}_{\mathcal{PCM}}(C.(M, P, \mu), \varepsilon A)) \\ &\cong H^{n-1}(\mathrm{Hom}_{\mathcal{G}rp}(kC.(M, P, \mu), A)) \\ &\cong H^n(P, A),\ n \geq 1. \end{aligned}$$

□

In [9], Gilbert introduced two invariants $\mathcal{H}_1(T, G, \partial)$ and $\mathcal{H}_2(T, G, \partial)$ for every crossed module (T, G, ∂), using the equivalence between the category of crossed modules and the category of group objects in the category of groupoids.

For each crossed module (T, G, ∂), Gilbert's homology groups are the abelian crossed modules

$$\mathcal{H}_1(T, G, \partial) = (T/\left[G, T\right], G/\left[G, G\right], \overline{\partial}),$$
$$\mathcal{H}_2(T, G, \partial) = (\Sigma, H_2(G), \sigma_{*|\Sigma}),$$

where $\Sigma = \mathrm{Ker}(\tau_*)$, and τ_* and σ_* are the homomorphisms induced in homology by the group homomorphisms

$$\tau\colon T\rtimes G \longrightarrow G, \quad \tau(t,g) = g,$$
$$\sigma\colon T\rtimes G \longrightarrow G, \quad \sigma(t,g) = \partial(t)g,$$

and $T\rtimes G$ denotes the semidirect product of G acting on T.

THEOREM 4.2. *If (T, G, ∂) is a crossed module, then*

$$H_1(T,G,\partial) \cong \mathcal{H}_1(T,G,\partial) \text{ and } H_2(T,G,\partial) \cong \mathcal{H}_2(T,G,\partial).$$

Proof. The first relation is clear because for each crossed module (T, G, ∂) we have $[T,T] = [\partial(T),T] \subset [G,T]$.

To prove the second relation note that if $(\overline{F}, F * F, \langle i_1, \mathrm{Id}\rangle_{|\overline{F}})$ is a free precrossed module then $\overline{F}\rtimes(F*F)$ is a free group since the short exact sequence

$$\overline{F} \rightarrowtail F*(F*F) \overset{\langle 0,\mathrm{Id}\rangle}{\twoheadrightarrow} F*F$$

is split, and the action induced by $F*F$ on $\overline{F}$ coincides with the action in the free precrossed module.

Thus, if $C.(T,G,\partial) = (M.,P.,\mu.) \twoheadrightarrow (T,G,\partial)$ is the standard free simplicial resolution of (T,G,∂), then $M.\rtimes P. \twoheadrightarrow T\rtimes G$ is a free simplicial resolution of the group $T\rtimes G$ since its underlying augmented simplicial set coincides with $\mathcal{U}(M.,P.,\mu.) \longrightarrow \mathcal{U}(M,P,\mu)$.

Let us take the group homomorphisms $\tau_n\colon M_n\rtimes P_n \longrightarrow P_n$, $\tau_n(m,p) = p$, and $\sigma_n\colon M_n\rtimes P_n \longrightarrow P_n$, $\sigma_n(m,p) = \mu_n(m)p$. It is easy to see that $\sigma.$ and $\tau.$ are morphisms of augmented simplicial groups

$$\begin{array}{ccc} M.\rtimes P. & \twoheadrightarrow & T\rtimes G \\ \sigma. \downdownarrows \tau. & & \sigma \downdownarrows \tau \\ P. & \twoheadrightarrow & G \end{array}$$

If we set

$$\mathcal{H}_2(T,G,\partial) = (\Sigma, H_2(G), (\sigma_1)_{*|\Sigma}),$$

then

$$H_2(T\rtimes G) = H_1((M.\rtimes P.)_{ab}), \quad \text{and} \quad \Sigma = \mathrm{Ker}((\tau_1)_*),$$

where

$$(\tau_1)_*, (\sigma_1)_*\colon H_1((M.\rtimes P.)_{ab}) \longrightarrow H_1((P.)_{ab}))$$

are induced in homology by $(\tau_1)_{ab}$ and $(\sigma_1)_{ab}$.

Since every epimorphism $M_n\rtimes P_n \overset{\tau_n}{\twoheadrightarrow} P_n$ is split, the short exact sequence of simplicial abelian precrossed modules

$$\mathrm{Ker}(\tau.)_{ab} \rightarrowtail (M.\rtimes P.)_{ab} \overset{(\tau.)_{ab}}{\twoheadrightarrow} (P.)_{ab}$$

is a weakly split sequence and the associated long exact sequence in homology provides the isomorphism

$$H_n(\mathrm{Ker}(\tau.)_{ab}) \cong \mathrm{Ker}((\tau.)_*\colon H_n((M.\rtimes P.)_{ab}) \longrightarrow H_n((P.)_{ab})).$$

It is easy to prove that $[M_n \rtimes P_n, M_n \rtimes P_n] = [M_n, M_n][P_n, M_n] \rtimes [P_n, P_n]$, so

$$(\tau_n)_{ab}\colon M_n/[M_n, M_n][P_n, M_n] \times P_n/[P_n, P_n] \longrightarrow P_n/[P_n, P_n]$$

is the projection to the second component, and we have a simplicial abelian group morphism

$$\begin{array}{ccc} \mathrm{Ker}(\tau.)_{ab} & \longrightarrow & \mathrm{Ker}(\tau_{ab}) \\ (\sigma.)_{ab}\downarrow & & \downarrow \sigma_{ab} \\ (G.)_{ab} & \longrightarrow & G_{ab} \end{array} \tag{1}$$

In homology, between the first two homology groups we obtain the abelian group homomorphism $\mathcal{H}_2(T, G, \partial)$.

Now observe that $(\sigma.)_{ab|\mathrm{Ker}(\tau.)_{ab}} = (\mu.)_{ab}$ and $\sigma_{ab|\mathrm{Ker}(\tau_{ab})} = \partial_{ab}$. It follows that the complex (1) is the abelianisation of the standard free simplicial resolution of (T, G, ∂), and so its first homology is also $H_2(T, G, \partial)$. □

Recall that the category of precrossed modules is equivalent to the category of simplicial groups of length 1 (*pre-cat*^{1}*-groups* in the notation of [4]), that is, a precrossed module (M, P, μ) is equivalent to the simplicial group

$$M \rtimes P \underset{\sigma}{\overset{\overset{i}{\curvearrowleft}}{\underset{\tau}{\rightrightarrows}}} P\,,$$

where $\tau(m, p) = p$ and $\sigma(m, p) = \mu(m)p$.

Following the proof of Theorem 4.2 and replacing the crossed module (T, G, ∂) by a precrossed module (M, P, μ), we deduce:

Theorem 4.3. *The nth homology group of a precrossed module (M, P, μ) is $H_n(M, P, \mu) = (\Sigma_n, H_n(P), \sigma_*)$, where $\Sigma_n = \mathrm{Ker}(\tau_*)$ and*

$$\tau_*, \sigma_*\colon H_n(M \rtimes P) \longrightarrow H_n(P)$$

are the group homomorphisms induced in homology by τ and σ.

Corollary 4.4. *If we regard a group G as the precrossed module $(G, 1, 1)$, then $H_n(G, 1, 1) = (H_n(G), 1, 1)$.*

Example. Let G be a group regarded as the precrossed module (G, G, Id). Then σ and τ composed with the isomorphism $G \times G \to G \rtimes G$ given by $(g_1, g_2) \mapsto (g_1 g_2^{-1}, g_2)$ become the projections from $G \times G$ to G. Thus Künneth' formula in the homology of groups and Theorem 4.3 yield

$$H_2(G, G, \mathrm{Id}) = (H_2(G) \oplus (H_1(G) \otimes H_1(G)), H_2(G), \sigma_*),$$

with σ_* acting as the identity on $H_2(G)$, and as zero on $H_1(G) \otimes H_1(G)$.

The category of crossed modules $\mathcal{CM}$ is a Birkhoff variety of $\mathcal{PCM}$. It is well known that the inclusion $I\colon \mathcal{CM} \longrightarrow \mathcal{PCM}$ has as left adjoint $\mathcal{P}\colon \mathcal{PCM} \longrightarrow \mathcal{CM}$, the Peiffer abelianisation which assigns to each precrossed module (M, P, μ) the quotient $(M/\langle M, M\rangle, P, \overline{\mu})$.

Consider the composition of adjunctions

$$\mathcal{S}et \underset{\mathcal{U}}{\overset{\mathcal{F}}{\rightleftarrows}} \mathcal{PCM} \underset{I}{\overset{\mathcal{P}}{\rightleftarrows}} \mathcal{CM}.$$

The tripleable forgetful functors $\mathcal{U}$ and $\mathcal{U}I$ provide the free cotriples (C, δ, ε) in $\mathcal{PCM}$ and $(\overline{C}, \delta, \varepsilon)$ in $\mathcal{CM}$. (The latter cotriple is the one obtained in [5].)

To relate our (co)homology theory of precrossed modules to the CCG -(co)homology theory of crossed modules we define natural transformations

$$r_n\colon C^n I \Rightarrow I\overline{C}^n$$

inductively by $r_{n+1} = r_n\overline{C} \circ C^n r_1$ for every $n \geq 1$, where r_1 is the natural transformation which sends each crossed module (T, G, ∂) to the canonical projection $C(T, G, \partial) \twoheadrightarrow \mathcal{P}C(T, G, \partial) = \overline{C}(T, G, \partial)$.

Lemma 4.5. *The transformations r_n commute with the face operators of the standard free resolutions $C.(T, G, \partial)$ and $\overline{C}.(T, G, \partial)$; that is, $r_n \circ \delta_i = \overline{\delta_i} \circ r_{n+1}$, where $\delta_i = C^{n-i}\delta C^i$ and $\overline{\delta_i} = \overline{C}^{n-i}\overline{\delta C}^i$.*

Proof. The proof is an easy calculation, given in [6] in a more general context. □

Thus, for every crossed module (T, G, ∂) we have the following surjective morphism of resolutions:

$$\begin{array}{ccccccccc}
\dots & C^n(T,G,\partial) & \dots & C^2(T,G,\partial) & \overset{\delta_1}{\underset{\delta_0}{\rightrightarrows}} & C(T,G,\partial) & \overset{\delta}{\twoheadrightarrow} & (T,G,\partial) \\
 & r_n \downarrow & & r_2 \downarrow & & r_1 \downarrow & & \parallel \\
\dots & \overline{C}^n(T,G,\partial) & \dots & \overline{C}^2(T,G,\partial) & \overset{\overline{\delta}_1}{\underset{\overline{\delta}_0}{\rightrightarrows}} & \overline{C}(T,G,\partial) & \overset{\overline{\delta}}{\twoheadrightarrow} & (T,G,\partial)
\end{array}$$

In the next theorem we denote the nth CCG-homology group of the crossed module (T, G, ∂) by $H_n^{CCG}(T, G, \partial)$ and the nth CCG-cohomology group of the crossed module (T, G, ∂) with coefficients in an abelian precrossed module (A, B, f) by $H^n_{CCG}((T, G, \partial), (A, B, f))$.

Theorem 4.6. *Let (T, G, ∂) be a crossed module and let (A, B, f) be an abelian precrossed module.*

(i) *We have*

$$H_1(T,G,\partial) = H_1^{CCG}(T,G,\partial),$$
$$H^1((T,G,\partial),(A,B,f)) = H^1_{CCG}((T,G,\partial),(A,B,f)).$$

(ii) *There exists a long exact sequence of abelian precrossed modules in homology*

$$\begin{aligned} \ldots H_n^{CCG}(T,G,\partial) &\longrightarrow H_{n-2}(\mathrm{Ker}(r.)_{ab}) \longrightarrow H_{n-1}(T,G,\partial) \\ &\longrightarrow H_{n-1}^{CCG}(T,G,\partial) \longrightarrow \ldots \longrightarrow H_1(\mathrm{Ker}(r.)_{ab}) \\ &\longrightarrow H_2(T,G,\partial) \longrightarrow H_2^{CCG}(T,G,\partial) \longrightarrow 0. \end{aligned}$$

(iii) *There exists a long exact sequence of abelian precrossed modules in cohomology*

$$\begin{aligned} 0 &\longrightarrow H^2_{CCG}((T,G,\partial),(A,B,f)) \longrightarrow H^2((T,G,\partial),(A,B,f)) \\ &\longrightarrow H^1(\mathrm{Coker}(\mathrm{Hom}_{\mathcal{PCM}}(r.,(A,B,f)))) \ldots H^n_{CCG}((T,G,\partial),(A,B,f)) \\ &\longrightarrow H^n((T,G,\partial),(A,B,f)) \longrightarrow H^{n-1}(\mathrm{Coker}(\mathrm{Hom}_{PCM}(r.,(A,B,f)))) \\ &\longrightarrow H^{n+1}_{CCG}((T,G,\partial),(A,B,f)) \longrightarrow \ldots \end{aligned}$$

Proof. Assertion (i) is clear since

$$H_1^{CCG}(T,G,\partial) = (T,G,\partial)_{ab}$$

and

$$H^1_{CCG}((T,G,\partial),(A,B,f)) = \mathrm{Hom}_{\mathcal{PCM}}((T,G,\partial),(A,B,f)).$$

To prove (ii) note that there exists a surjective morphism of complexes

$$\begin{array}{ccccccccc} \ldots & C^n(T,G,\partial)_{ab} & \ldots & C^2(T,G,\partial)_{ab} & \xrightarrow{\partial_2} & C(T,G,\partial)_{ab} & \longrightarrow & 0 \\ & (r_n)_{ab}\downarrow & & (r_2)_{ab}\downarrow & & (r_1)_{ab}\downarrow & & \\ \ldots & \overline{C}^n(T,G,\partial)_{ab} & \ldots & \overline{C}^2(T,G,\partial)_{ab} & \xrightarrow{\overline{\partial}_2} & \overline{C}(T,G,\partial)_{ab} & \longrightarrow & 0 \end{array}$$

and that $(r_1)_{ab}$ is the identity map because the abelianisation (as crossed module) of the Peiffer abelianisation of a precrossed module is exactly the abelianisation of the precrossed module. Hence the sequence follows from the long exact sequence in homology associated with the short exact sequence of complexes

$$\mathrm{Ker}(r.)_{ab} \rightarrowtail C.(T,G,\partial)_{ab} \overset{(r.)_{ab}}{\twoheadrightarrow} \overline{C}.(T,G,\partial)_{ab}$$

Part (iii) follows similarly, using

$$\begin{aligned} \mathrm{Hom}_{\mathcal{PCM}}(C(T,G,\partial),(A,B,f)) &= \mathrm{Hom}_{\mathcal{PCM}}((C(T,G,\partial))_{ab},(A,B,f)) \\ &= \mathrm{Hom}_{\mathcal{PCM}}(\overline{C}(T,G,\partial),(A,B,f)). \qquad \square \end{aligned}$$

Remark 2. For $n \geq 2$, we have

$$H_n(G,G,\mathrm{Id}) \neq H_n^{CCG}(G,G,\mathrm{Id}) = (H_n(G),H_n(G),\mathrm{Id}).$$

REMARK 3. The above results and Theorem 4.3 show that the low-dimensional homology given by Gilbert [9] is not one of crossed modules, but of precrossed modules.

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