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Baer invariants and cohomology of precrossed modules

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Abstract In this paper we study Baer invariants of precrossed modules relative to the subcategory of crossed modules, following Fröhlich and Furtado-Coelho's general theory on Baer invariants in varieties of Ω-groups and Modi's theory on higher dimensional Baer invariants.

Several homological invariants of precrossed and crossed modules were defined in the last two decades. We show how to use Baer invariants in order to connect these various homology theories.

First, we express the low-dimensional Baer invariants of precrossed modules in terms of a new non-abelian tensor product of a precrossed module. This expression is used to analyze the connection between the Baer invariants and the homological invariants of precrossed modules defined by Conduché and Ellis. Specifically we prove that the second homological invariant of Conduché and Ellis is in general a quotient of the first component of the Baer invariant we consider.

The definition of classical Baer invariants is generalized using homological methods. These generalized Baer invariants of precrossed modules are applied to the construction of five term exact sequences connecting the generalized Baer invariants with the cohomology theory of crossed modules considered by Carrasco, Cegarra and R.-Grandjeán and the cohomology theory of precrossed modules.

Keywords Baer invariant · Precrossed module · Crossed module · Non-abelian tensor product · Cohomology · Central extension

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1 Introduction

In [21, 22], Fröhlich and Furtado-Coelho establish a general theory on Baer invariants of Ω-groups. More recently, Fröhlich's and Furtado-Coelho's work was extended by Everaert and Van der Linden in [18] to the context of semi-abelian categories.

Several Baer invariants of precrossed modules can be constructed relative to a *Birkhoff subcategory* of the category of precrossed modules $\mathcal{PCM}$, that is, a reflective subcategory of $\mathcal{PCM}$ which is full and closed in $\mathcal{PCM}$ under regular quotients and subobjects [25]. Such Birkhoff subcategories are just subvarieties.

In this paper we construct Baer invariants of precrossed modules relative to the subcategory of crossed modules $\mathcal{CM}$, following Fröhlich and Furtado-Coelho's general theory on Baer invariants and Modi's theory of higher dimensional Baer invariants in varieties of Ω-groups [30].

Several homological invariants of precrossed and crossed modules were defined in the last two decades [2, 9, 13, 20, 31]. We will see how to use Baer invariants in order to show the connection between the aforementioned homological invariants.

The text is organized as follows. In Section 2 we briefly recall some of the basic definitions and results of precrossed modules; in particular, the Peiffer subgroups $\langle M, M\rangle$ and $\langle N, N'\rangle$ of a precrossed module (M, P, μ) and of two precrossed submodules (N, Q, μ) and (N', Q', μ) of (M, P, μ) are respectively defined.

The low dimensional Baer invariants $\Delta V(M, P, \mu)$ and $DV(M, P, \mu)$ of a precrossed module (M, P, μ) are precrossed modules. They are defined in Section 3 in terms of Peiffer subgroups of a projective presentation $(K, R, \tau) \rightarrowtail (W, F, \tau) \overset{\pi}{\twoheadrightarrow} (M, P, \mu)$ of (M, P, μ). Both $\Delta V(M, P, \mu)$ and $DV(M, P, \mu)$ have a trivial second component and a non-trivial first component $\Delta_1 V(M, P, \mu) = \frac{K\cap\langle W,W\rangle}{\langle W,K\rangle}$ and $D_1V(M, P, \mu) = \frac{\langle W,W\rangle}{\langle W,K\rangle}$.

These first components of the invariants can be expressed in terms of a non-abelian tensor product of groups. For a precrossed module (M, P, μ), we define in Section 4 a group $M \otimes^P M$ isomorphic to $D_1V(M, P, \mu)$, and such that there is a canonical group homomorphism $\kappa_M \colon M \otimes^P M \to M$, whose kernel is isomorphic to $\Delta_1 V(M, P, \mu)$. We prove that the Conduché and Ellis [13] homological invariant $H_2(M)_P$ of a precrossed module (M, P, μ) is strongly connected to the Baer invariants defined in this paper, since it is in general a quotient of $\Delta_1 V(M, P, \mu)$.

In the last section we generalize the classical construction of the previous Baer invariants following [30]. In his thesis, Modi defines higher dimensional Baer invariants in a variety of Ω-groups relative to a subvariety, using Keune's homotopical theory [28].

The nth *Baer invariant* $\mathcal{B}_n(M, P, \mu)$ of a precrossed module (M, P, μ) relative to the variety of crossed modules is defined as the value of the nth left derived functor of the Peiffer abelianization in (M, P, μ). These higher dimensional Baer invariants are the homology invariants with coefficients in the subvariety of crossed modules. The first of these invariants $\mathcal{B}_1(M, P, \mu)$ coincides with $\Delta V(M, P, \mu)$.

The higher dimensional Baer invariants $\mathcal{B}_n(M,P,\mu)$ of precrossed modules relative to the variety of crossed modules play a fundamental role in the study of the relationship between the cohomology theories defined in the varieties of Ω-groups $\mathcal{PCM}$ and $\mathcal{CM}$. We obtain five term exact sequences connecting the cohomology groups H^n and $H^n_{\mathcal{CM}}$ of precrossed and crossed modules defined respectively in [2] and [9] as instances of Barr and Beck's general theory [3].

As an application, we obtain some conditions under which the group of congruence classes of central extensions in $\mathcal{PCM}$ of a precrossed module (M,P,μ) by an abelian precrossed module (A,B,f) coincides with the group of congruence classes of central extensions in $\mathcal{CM}$ of the Peiffer abelianization $(M/\langle M,M\rangle,P,\mu)$ by (A,B,f).

2 Preliminaries on Precrossed Modules

In this section we recall some standard results on precrossed modules which will be used in the sequel. Details can be found in [2].

A *precrossed module* (M,P,μ) is a group homomorphism $\mu\colon M\to P$ together with an action of P on M, denoted ${}^pm$ for $m\in M$ and $p\in P$, such that $\mu({}^pm)=p\mu(m)p^{-1}$ for all $m\in M$ and $p\in P$. (M,P,μ) is called a *crossed module* if in addition it satisfies the Peiffer identity ${}^{\mu(m)}m'=mm'm^{-1}$ for every $m,m'\in M$.

A *precrossed (crossed) module morphism* $\psi=(\psi_1,\psi_2)\colon (Y,X,\delta)\to(M,P,\mu)$ between two precrossed (crossed) modules (Y,X,δ) and (M,P,μ), is a pair of group homomorphisms $\psi_1\colon Y\to M$ and $\psi_2\colon X\to P$ such that $\mu\psi_1=\psi_2\delta$ and $\psi_1({}^xy)={}^{\psi_2(x)}\psi_1(y)$ for $y\in Y$ and $x\in X$.

We denote by $\mathcal{PCM}$ the category of precrossed modules, and by $\mathcal{CM}$ the subcategory of crossed modules.

The category $\mathcal{PCM}$ is equivalent to a variety of Ω-groups: the variety $\mathcal{X}$ of groups with operators $\Omega=\{1,i,s,t,\cdot\}$, where 1 is the 0-ary operation unit, i,s,t are 1-ary operators, with i the inversion operator, and $\cdot$ is the 2-ary operation of the group, satisfying the relations

$$s(1)=1,\quad t(1)=1,\quad s(xy)=s(x)s(y),\quad t(xy)=t(x)t(y),\tag{1}$$
$$t\circ s=s\circ s=s,\qquad s\circ t=t\circ t=t,\tag{2}$$

where (1) implies that the operations s and t are group homomorphisms $X\to X$, for any $X\in\mathcal{X}$.

With any Ω-group $(X,s,t)\in\mathcal{X}$ is associated the precrossed module

$$\Phi(X,s,t)=\big(\mathrm{Ker}(s),\mathrm{Im}(s),t_{|\mathrm{Ker}(s)}\big),$$

where the action is given by conjugation. Conversely, with a precrossed module (M,P,μ) is associated the Ω-group

$$\Psi(M,P,\mu)=(M\rtimes P,s,t)\,,$$

where $s(m,p)=(1,p)$ and $t(m,p)=(1,\mu(m)p)$.

Under this equivalence $\Psi\colon\mathcal{PCM}\to\mathcal{X}$, the subcategory $\mathcal{CM}$ of crossed modules corresponds to the Birkhoff subvariety of all Ω-groups which satisfy (1), (2) and $[\mathrm{Ker}(t),\mathrm{Ker}(s)]=1$ (see [29] and [26]).

A *presentation* or an *extension* of a precrossed module (M, P, μ) is a regular epimorphism $(\psi_1, \psi_2)\colon (Y, X, \delta) \twoheadrightarrow (M, P, \mu)$. It can be easily checked that these are precrossed module morphisms in which both of the components ψ_1 and ψ_2 are surjective group homomorphisms.

A *precrossed submodule* (N, Q, μ') of a precrossed module (M, P, μ) is a subobject of (M, P, μ) in $\mathcal{PCM}$. Equivalently, it is a precrossed module such that N and Q are, respectively, subgroups of M and P, the action of Q on N is induced by the one of P on M and $\mu' = \mu_{|N}$. If (N, Q, μ') is a normal subobject (i.e., the kernel of some morphism) of (M, P, μ) then we say that it is a *normal precrossed submodule*. This means that besides N and Q are normal in M and P, ${}^{p}n \in N$ and ${}^{q}mm^{-1} \in N$ for every $p \in P$, $q \in Q$, $m \in M$ and $n \in N$.

For a normal precrossed submodule (N, Q, μ) of (M, P, μ), the quotient precrossed module $(M, P, \mu)/(N, Q, \mu)$ is $(M/N, P/Q, \overline{\mu})$, where the homomorphism $\overline{\mu}$ is induced by μ and P/Q acts on M/N by ${}^{pQ}mN = ({}^{p}m)N$ for $p \in P$ and $m \in M$.

The kernel of a precrossed module morphism $(\psi_1, \psi_2)\colon (Y, X, \delta) \to (M, P, \mu)$ is the normal precrossed submodule $(\operatorname{Ker}(\psi_1), \operatorname{Ker}(\psi_2), \delta)$ of (Y, X, δ). Its image is the precrossed submodule $(\operatorname{Im}(\psi_1), \operatorname{Im}(\psi_2), \mu)$ of (M, P, μ).

Given a precrossed module (M, P, μ), the *Peiffer commutator* of two elements $m_1, m_2 \in M$ is $\langle m_1, m_2\rangle = m_1 m_2 m_1^{-1}\, {}^{\mu(m_1)}m_2^{-1}$. We call *Peiffer subgroup* of two precrossed submodules (N, Q, μ) and (N', Q', μ') of a precrossed module (M, P, μ) the subgroup $\langle N, N'\rangle$ of M generated by the Peiffer elements $\langle n, n'\rangle$ and $\langle n', n\rangle$ with $n \in N$ and $n' \in N'$. For basic properties of these groups see [13].

The Peiffer subgroup $\langle M, M\rangle$ of a precrossed module (M, P, μ) is the smallest normal subgroup of M such that the quotient $(M/\langle M, M\rangle, P, \mu)$ is a crossed module. We can define a functor $U\colon \mathcal{PCM} \to \mathcal{CM}$, which assigns to a precrossed module (M, P, μ) the quotient $U(M, P, \mu) = (M, P, \mu)/(\langle M, M\rangle, 1, 1)$. The functor U is left adjoint to the inclusion functor $i\colon \mathcal{CM} \to \mathcal{PCM}$, and is usually called the *Peiffer abelianization functor*,

$$\begin{array}{ccc} \mathcal{PCM} & (M, P, \mu) & (T, G, \partial) \\ U \downarrow\uparrow i & \downarrow & \uparrow \\ \mathcal{CM} & (M/\langle M, M\rangle, P, \mu) & (T, G, \partial). \end{array}$$

An *abelian* precrossed module is an abelian object in the category $\mathcal{PCM}$, that is, a precrossed module which admits an internal abelian group structure. It is easy to check that an abelian precrossed module is exactly a precrossed module (A, B, f) such that A and B are abelian groups and B acts trivially on A (see [2] and [6]).

Given a precrossed module (M, P, μ), an (M, P, μ)-*module* in $\mathcal{PCM}$ is an abelian group object in the comma category $\mathcal{PCM}/(M, P, \mu)$ [5]. Equivalently, an (M, P, μ)-module is an abelian precrossed module (A, B, f) together with a split short exact sequence in $\mathcal{PCM}$

$$(A, B, f) \rightarrowtail (Y, X, \delta) \underset{\leftarrow}{\twoheadrightarrow} (M, P, \mu),$$

since the category $\mathcal{PCM}$ is strongly protomodular [7].

The split exact sequence $(A, B, f) \rightarrowtail (Y, X, \delta) \underset{\leftarrow}{\twoheadrightarrow} (M, P, \mu)$ is equivalent to a split exact sequence $A \times B \rightarrowtail Y \rtimes X \underset{\leftarrow}{\twoheadrightarrow} M \rtimes P$ of Ω-groups, which is determined, up to isomorphism, by the Ω-groups $A \times B$ and $M \rtimes P$ and the induced group action of $M \rtimes P$ on $A \times B$.

Remark 1 From the split exact sequence $A \times B \rightarrowtail Y \rtimes X \rightleftarrows M \rtimes P$ of Ω-groups we deduce that the precrossed module (Y, X, δ) is equivalent to the Ω-group $Y \rtimes X \cong (A \times B) \rtimes (M \rtimes P)$ with operators $s, t\colon (A \times B) \rtimes (M \rtimes P) \to (A \times B) \rtimes (M \rtimes P)$ defined by $s\big((a, b), (m, p)\big) = \big((1, b), (1, p)\big)$ and $t\big((a, b), (m, p)\big) = \big((1, f(a)b), (1, \mu(m)p)\big)$. Since $\mathrm{Ker}(s) = (A \times 1) \rtimes (M \rtimes 1) \cong A \rtimes M$ and $\mathrm{Im}(s) = (1 \times B) \rtimes (1 \rtimes P) \cong B \rtimes P$, the precrossed module (Y, X, δ) is isomorphic to the precrossed module $(A \rtimes M, B \rtimes P, \{f, \mu\})$, with action induced by conjugation in $(A \times B) \rtimes (M \rtimes P)$.

An (M, P, μ)-*module morphism* is a commutative diagram

$$\begin{tikzcd}
(A, B, f) \arrow[r, tail] \arrow[d, "{(\psi_1,\psi_2)}"'] & (Y, X, \delta) \arrow[r, shift left, two heads] \arrow[d] & (M, P, \mu) \arrow[l, shift left] \arrow[d, equal] \\
(A', B', f') \arrow[r, tail] & (Y', X', \delta') \arrow[r, shift left, two heads] & (M, P, \mu) \arrow[l, shift left]
\end{tikzcd}$$

or equivalently a morphism $\psi = \{\psi_1, \psi_2\}\colon A \times B \to A' \times B'$ of Ω-groups which is compatible with the $(M \rtimes P)$-action (see [11]).

3 Construction of the Baer invariants

The Birkhoff subcategory $\mathcal{CM} \subset \mathcal{PCM}$ of crossed modules corresponds with a Birkhoff subfunctor $V\colon \mathcal{PCM} \to \mathcal{PCM}$ of the identity functor $1_{\mathcal{PCM}}$, which sends a precrossed module (M, P, μ) to the Peiffer subgroup $V(M, P, \mu) = (\langle M, M\rangle, 1, 1)$.

In this section we will give a first expression of the classical Baer invariants ΔV and DV associated to this Birkhoff subfunctor V.

Denote by $\mathrm{Pr}\mathcal{PCM}$ the category of presentations of precrossed modules, where a morphism $(f_0, f)\colon p \to p'$ is a commutative diagram

$$\begin{tikzcd}
(Y, X, \delta) \arrow[r, two heads, "p"] \arrow[d, "f_0"'] & (M, P, \mu) \arrow[d, "f"] \\
(Y', X', \delta') \arrow[r, two heads, "p'"] & (M', P', \mu')\,.
\end{tikzcd}$$

Recall from [18] that a functor $B\colon \mathrm{Pr}\mathcal{PCM} \to \mathcal{PCM}$ is called a *Baer invariant* if for each pair of morphisms $(f_0, f), (g_0, g)\colon p \to p'$ with $f = g$, it is satisfied that $B(f_0, f) = B(g_0, g)$. From this way we obtain functors which associate isomorphic precrossed modules to “similar” presentations [18, Proposition 3.4].

V induces a functor $V_0\colon \mathrm{Pr}\mathcal{PCM} \to \mathcal{PCM}$ by putting

$$V_0\big((Y, X, \delta) \overset{p}{\twoheadrightarrow} (M, P, \mu)\big) = V(Y, X, \delta)$$

and a functor $V_1\colon \mathrm{Pr}\mathcal{PCM} \to \mathcal{PCM}$, which is the minimum of the functors $S\colon \mathrm{Pr}\mathcal{PCM} \to \mathcal{PCM}$ satisfying

(i) S is a normal subfunctor of V_0 (i.e. $S(p) \to V_0(p)$ is a kernel);

(ii) $V(R[p]) \subset R\big[V(Y,X,\delta) \twoheadrightarrow \frac{V(Y,X,\delta)}{S(p)}\big]$ for each presentation $(Y,X,\delta) \overset{p}{\twoheadrightarrow} (M,P,\mu)$, where $R[f]$ denotes the kernel pair of f [18, Proposition 3.8].

The following description of V_1 is a special case of a result in [15]:

Proposition 1 *[15, Proposition 2.3] Let $(N,Q,\delta) \rightarrowtail (Y,X,\delta) \overset{p}{\twoheadrightarrow} (M,P,\mu)$ be a presentation of the precrossed module (M,P,μ). Then $V_1(p) = (\langle Y,N\rangle,1,1)$.*

There are Baer invariants $\Delta V, DV\colon \mathrm{Pr}\mathcal{PCM} \to \mathcal{PCM}$ defined for a presentation $(N,Q,\delta) \rightarrowtail (Y,X,\delta) \overset{p}{\twoheadrightarrow} (M,P,\mu)$ as in [18, Proposition 4.6]

$$\Delta V(p) = \frac{(N,Q,\delta)\cap V(Y,X,\delta)}{V_1(p)}$$

and

$$DV(p) = \frac{V(Y,X,\delta)}{V_1(p)}.$$

However we will only be interested in the restriction of the Baer invariants $B\colon \mathrm{Pr}\mathcal{PCM} \to \mathcal{PCM}$ to the subcategory of projective presentations of precrossed modules. This restriction gives rise to functors $B\colon \mathcal{PCM} \to \mathcal{PCM}$ which are independent of the choice of the projective presentation [18, Proposition 3.18].

So there is an exact (pointwise) sequence of functors $\mathcal{PCM} \to \mathcal{PCM}$ [18, Remark 4.10]

$$0 \Rightarrow \Delta V \Rightarrow DV \Rightarrow V \Rightarrow 0$$

where ΔV is defined for a precrossed module (M,P,μ), with projective presentation $(K,R,\tau) \rightarrowtail (W,F,\tau) \overset{\pi}{\twoheadrightarrow} (M,P,\mu)$, as

$$\Delta V(M,P,\mu) = \left(\frac{K\cap\langle W,W\rangle}{\langle W,K\rangle},1,1\right), \tag{3}$$

while $DV(M,P,\mu)$ is

$$DV(M,P,\mu) = \left(\frac{\langle W,W\rangle}{\langle W,K\rangle},1,1\right).$$

We will denote the first component of $\Delta V(M,P,\mu)$ by $\Delta_1 V(M,P,\mu)$ and the first component of $DV(M,P,\mu)$ by $D_1V(M,P,\mu)$.

Remark 2 For every projective precrossed module (W,F,τ), the Baer invariant $\Delta V(W,F,\tau) = 0$, while $DV(W,F,\tau) = V(W,F,\tau)$.

The following five term exact sequence follows from [18, Theorem 5.9]. It is the tail of a long exact homology sequence [14] (see also [23]), whose terms can also be made fully explicit [17, Section 9.5].

Theorem 1 *[18, Theorem 5.9] For each extension of precrossed modules $(N,Q,\delta) \rightarrowtail (Y,X,\delta) \overset{p}{\twoheadrightarrow} (M,P,\mu)$ there is an exact sequence of crossed modules*

$$\Delta V(Y,X,\delta) \xrightarrow{\Delta V(p)} \Delta V(M,P,\mu) \longrightarrow \frac{(N,Q,\delta)}{V_1(p)} \longrightarrow U(Y,X,\delta) \overset{U(p)}{\twoheadrightarrow} U(M,P,\mu)\,.$$

Remark 3 The first component of this exact sequence is an exact sequence of groups

$$\varDelta_1 V(Y,X,\delta) \longrightarrow \varDelta_1 V(M,P,\mu) \longrightarrow \tfrac{N}{\langle Y,N\rangle} \longrightarrow \tfrac{Y}{\langle Y,Y\rangle} \twoheadrightarrow \tfrac{M}{\langle M,M\rangle}\,.$$

Remark 4 In [13], Conduché and Ellis define the second homology group $H_2(M)_P$ of a precrossed module (M,P,μ) by means of a Hopf type formula similar to the expression for $\varDelta_1 V(M,P,\mu)$. They take a free presentation $1 \to R \to F \to M \to 1$ of M in the category $\mathcal{PCM}_P$ of precrossed modules with fixed action group P, and define

$$H_2(M)_P = \frac{R \cap \langle F,F\rangle}{\langle F,R\rangle}\,.$$

This homology coincides with the Baer invariants in the category $\mathcal{PCM}_P$ relative to the multi-sorted variety $\mathcal{CM}_P$ of crossed modules with fixed action group P defined in [20]. $\mathcal{PCM}_P$ is a multi-sorted variety which is not a one-sorted variety (see [16, Proposition 2.1]).

In general, $H_2(M)_P$ is a quotient of $\varDelta_1 V(M,P,\mu)$:

Proposition 2 *For a precrossed module (M,P,μ) there is a natural regular epimorphism of groups $\varDelta_1 V(M,P,\mu) \twoheadrightarrow H_2(M)_P$.*

Proof For a free presentation $1 \to R \to F \xrightarrow{\varphi} M \to 1$ of M in $\mathcal{PCM}_P$ there is an extension of precrossed modules $(R,1,1) \rightarrowtail (F,P,\mu\varphi) \twoheadrightarrow (M,P,\mu)$. From Theorem 1 we get a natural exact sequence of groups

$$\varDelta_1 V(F,P,\mu\varphi) \longrightarrow \varDelta_1 V(M,P,\mu) \longrightarrow \tfrac{R}{\langle F,R\rangle} \longrightarrow \tfrac{F}{\langle F,F\rangle} \twoheadrightarrow \tfrac{M}{\langle M,M\rangle}$$

and $H_2(M)_P$ is the kernel of $\frac{R}{\langle F,R\rangle} \to \frac{F}{\langle F,F\rangle}$. □

4 Tensor product of a precrossed module

In this section we introduce the tensor product of a precrossed module. It will be useful for giving computable expressions of the Baer invariants $\varDelta V(M,P,\mu)$ and $DV(M,P,\mu)$.

For a precrossed module (M,P,μ) we define $M \otimes^P M$ as the group with generators $m \otimes m'$, for $m, m' \in M$ and with the following four relations

$$m \otimes m'm'' = (m \otimes m')(m \otimes m'')(\langle m,m''\rangle^{-1} \otimes {}^{\mu(m)}m') \tag{4}$$

$$mm' \otimes m'' = (m \otimes m'm''m'^{-1})({}^{\mu(m)}m' \otimes {}^{\mu(m)}m'') \tag{5}$$

$$\langle m,m'\rangle \otimes \langle n,n'\rangle = (m \otimes m')(n \otimes n')(m \otimes m')^{-1}(n \otimes n')^{-1} \tag{6}$$

$$(\langle m,m'\rangle \otimes m'')(m'' \otimes \langle m,m'\rangle) = (m \otimes m')({}^{\mu(m'')}m \otimes {}^{\mu(m'')}m')^{-1} \tag{7}$$

where $m, m', m'', n, n' \in M$.

Remark 5 In [13], Conduché and Ellis define a group $M \wedge_P M$ with generators $m \wedge m'$ subject to the relations (4)–(7) besides the relation

$$k \wedge k = 1$$

for $k \in \mathrm{Ker}(\mu)$. So there is a canonical regular epimorphism $M \otimes^P M \twoheadrightarrow M \wedge_P M$.

Definition 1 A precrossed module (M, P, μ) is said to be *aspherical* if $\operatorname{Ker}(\mu) = \langle M, M\rangle$ (see [12] for the topological motivation).

Lemma 1 *If (M, P, μ) is an aspherical precrossed module then $M \otimes^P M \cong M \wedge_P M$.*

Proof Note that for an element $k \in \operatorname{Ker}(\mu)$, if $k \otimes k = 1$ then $k^{-1} \otimes k^{-1} = 1$: relation (4) implies $1 = k \otimes kk^{-1} = (k \otimes k)(k \otimes k^{-1})(\langle k, k^{-1}\rangle^{-1} \otimes {}^{\mu(k)}k) = k \otimes k^{-1}$, and $1 = kk^{-1} \otimes k^{-1} = (k \otimes k^{-1}k^{-1}k)({}^{\mu(k)}k^{-1} \otimes {}^{\mu(k)}k^{-1}) = k^{-1} \otimes k^{-1}$ by (5).

$k \in \operatorname{Ker}(\mu)$ can be written $k = k_1 \cdots k_n$, where each k_i is an element of the form $\langle m_i, m_i'\rangle$ or of the form $\langle m_i, m_i'\rangle^{-1}$. Relation (6) implies that $\langle m_i, m_i'\rangle \otimes \langle m_i, m_i'\rangle = 1$, so $k_i \otimes k_i = 1$.

We will prove by induction in n that $k \otimes k = 1$ holds in $M \otimes^P M$: supposing that the result is true for every product $k = k_1 \cdots k_n$, we will verify that the same occurs to the elements kk_{n+1} and kk_{n+1}^{-1}, where $k_{n+1} = \langle m_{n+1}, m_{n+1}'\rangle$: by (5), $k_{n+1}k \otimes k_{n+1}k = (k_{n+1} \otimes kk_{n+1}kk^{-1})({}^{\mu(k_{n+1})}k \otimes {}^{\mu(k_{n+1})}k_{n+1}k) = (k_{n+1} \otimes kk_{n+1})(k \otimes k_{n+1}k)$, and by relation (4), $(k_{n+1} \otimes kk_{n+1})(k \otimes k_{n+1}k) = (k_{n+1} \otimes k)(k_{n+1} \otimes k_{n+1})(\langle k_{n+1}, k_{n+1}\rangle^{-1} \otimes {}^{\mu(k_{n+1})}k)(k \otimes k_{n+1})(k \otimes k)(\langle k, k\rangle^{-1} \otimes {}^{\mu(k)}k_{n+1}) = (k_{n+1} \otimes k)(k \otimes k_{n+1})$ which is zero by (6).

Analogously $kk_{n+1} \otimes kk_{n+1} = (k \otimes k_{n+1})(k_{n+1} \otimes k) = 1$.

Finally note that $kk_{n+1}^{-1} \otimes kk_{n+1}^{-1} = 1$ since $k_{n+1}k^{-1} \otimes k_{n+1}k^{-1} = 1$. □

Lemma 2 *Every projective precrossed module is aspherical.*

Proof The Peiffer abelianization $\left(\frac{W}{\langle W, W\rangle}, F, \overline{\tau}\right)$ of a projective precrossed module (W, F, τ) is a projective crossed module. $\overline{\tau}$ is injective since every projective crossed module is a retract of a free crossed module, and free crossed modules are injections [9, Proposition 3]. Then $\operatorname{Ker}(\tau) = \langle W, W\rangle$. □

For any precrossed module (M, P, μ) we will denote by $\kappa_M \colon M \otimes^P M \to M$ the group homomorphism defined by $\kappa_M(m \otimes m') = \langle m, m'\rangle$. It is the composition of the canonical epimorphism $M \otimes^P M \twoheadrightarrow M \wedge_P M$ with the group homomorphism $\partial_2 \colon M \wedge_P M \to M$ defined in [13] by $\partial_2(m \wedge m') = \langle m, m'\rangle$.

Lemma 3 *For every free precrossed module (W, F, τ) the homomorphism κ_W establishes an isomorphism $W \otimes^F W \cong \langle W, W\rangle$.*

Proof The homomorphism $W \wedge_F W \to W$ defined by $w \wedge w' \mapsto \langle w, w'\rangle$ is injective [13] (see also [4]) since every free precrossed module (W, F, τ) is a free F-precrossed module in the sense of [4] with F a free group.

Every free precrossed module is projective, so $W \otimes^F W \cong W \wedge_F W$ by Lemmas 1 and 2. □

Theorem 2 *For every precrossed module (M, P, μ) there is an isomorphism of groups $D_1V(M, P, \mu) \cong M \otimes^P M$.*

Proof Consider a free presentation $(K, R, \tau) \rightarrowtail (W, F, \tau) \xrightarrow{(\pi_1, \pi_2)} (M, P, \mu)$ of (M, P, μ). Denote by L_W and L_M the free groups generated by the sets $W \times W$ and $M \times M$ respectively, and by $L_{\pi_1} \colon L_W \twoheadrightarrow L_M$ the induced epimorphism defined in generators by $L_{\pi_1}(w,w') = \big(\pi_1(w),\pi_1(w')\big)$. There is a commutative diagram of group epimorphisms

$$\begin{tikzcd} L_W \arrow[r, two heads, "L_{\pi_1}"] \arrow[d, two heads, "\theta_W"'] & L_M \arrow[d, two heads, "\theta_M"] \\ W \otimes^F W \arrow[r, two heads, "\otimes^{\pi_1}"] & M \otimes^P M \end{tikzcd}$$

where θ_W (and similarly θ_M) is defined by $\theta_W(w,w') = w \otimes w'$, while $\otimes^{\pi_1}$ is defined by $\otimes^{\pi_1}(w \otimes w') = \big(\pi_1(w) \otimes \pi_1(w')\big)$.

It is easy to verify that the homomorphism L_{π_1} induces an epimorphism in the kernels $\mathrm{Ker}(\theta_W) \twoheadrightarrow \mathrm{Ker}(\theta_M)$ and then $\mathrm{Ker}(\otimes^{\pi_1}) = \theta_W\big(\mathrm{Ker}\,\big(L_{\pi_1}\big)\big)$. It is also easy to check that $\mathrm{Ker}(L_{\pi_1})$ is the normal subgroup of L_W generated by the elements $(w_1,w_2)(w_1',w_2')^{-1}$ such that $\pi_1(w_1) = \pi_1(w_1')$ and $\pi_1(w_2) = \pi_1(w_2')$, for $w_1,w_2,w_1',w_2' \in W$. Its image by θ_W in $W \otimes^F W$ is the normal subgroup generated by the elements $(w_1 \otimes w_2)(w_1' \otimes w_2')^{-1}$ such that $\pi_1(w_1) = \pi_1(w_1')$ and $\pi_1(w_2) = \pi_1(w_2')$, which by the formulas (4), (5) and (7) coincides with the normal subgroup of $W \otimes^F W$ generated by the elements $w \otimes k$ and $k \otimes w$ for $w \in W$ and $k \in K$. Thus the image of this subgroup by the isomorphism $W \otimes^F W \cong \langle W,W\rangle$ is $\langle W,K\rangle$, and then $M \otimes^P M \cong \frac{Y \otimes^F Y}{\mathrm{Ker}(\otimes^{\pi_1})} \cong \frac{\langle W,W\rangle}{\langle W,K\rangle} \cong D_1V(M,P,\mu)$. □

Theorem 3 *There is an isomorphism* $\Delta_1 V(M,P,\mu) \cong \mathrm{Ker}(M \otimes^P M \xrightarrow{\kappa_M} M)$ *for every precrossed module* (M,P,μ).

Proof In the conditions of Theorem 2, from the commutative diagram

$$\begin{tikzcd} W \otimes^F W \arrow[r, two heads, "\otimes^{\pi_1}"] \arrow[d, two heads, "\pi_1 \circ \kappa_W"'] & M \otimes^P M \arrow[d, two heads, "\kappa_M"] \\ \langle M,M\rangle \arrow[r, equal] & \langle M,M\rangle \end{tikzcd}$$

we deduce that $\mathrm{Ker}(\kappa_M)$ is isomorphic to $\frac{\mathrm{Ker}(\pi_1 \circ \kappa_W)}{\mathrm{Ker}(\otimes^{\pi_1})}$. But from Lemma 3 the images of $\mathrm{Ker}(\pi_1 \circ \kappa_W)$ and $\mathrm{Ker}(\otimes^{\pi_1})$ by the isomorphism $W \otimes^F W \cong \langle W,W\rangle$ are respectively $K \cap \langle W,W\rangle$ and $\langle W,K\rangle$. □

Corollary 1 *There is an isomorphism* $\Delta_1 V(M,P,\mu) \cong \Delta_1 V(M,\mu(M),\mu)$ *for every precrossed module* (M,P,μ).

Proof It follows from the isomorphism $M \otimes^{\mu(M)} M \cong M \otimes^P M$ (it can also be obtained directly through the Hopf formula (3) for ΔV). □

Remark 6 In [13, Théorème 2] it is shown that for a precrossed module (M,P,μ) which allows a free presentation $1 \to R \to F \to M \to 1$ in $\mathcal{PCM}_P$ satisfying that the homomorphism $F \wedge_P F \to F$ is injective, the homology group $H_2(M)_P$ is isomorphic to $\mathrm{Ker}(M \wedge_P M \to M)$. This happens for example when P is a free group.

Corollary 2 *Let* (M,P,μ) *be an aspherical precrossed module with* $\mu(M)$ *a free group. Then* $\Delta_1 V(M,P,\mu) \cong H_2(M)_P$.

Proof Take a free presentation $1 \to R \to F \to M \to 1$ of M in $\mathcal{PCM}_P$. By [4, Lemma 7.15] it is also a free presentation of M in $\mathcal{PCM}_{\mu(M)}$.

By definition, $H_2(M)_P = \frac{R \cap \langle F,F\rangle}{\langle F,R\rangle} = H_2(M)_{\mu(M)}$, and by Remark 6 this group is isomorphic to $\operatorname{Ker}(M \wedge_{\mu(M)} M \to M)$. By Lemma 1, $M \wedge_{\mu(M)} M \cong M \otimes^{\mu(M)} M$, so $H_2(M)_P \cong \operatorname{Ker}(M \wedge_{\mu(M)} M \to M) \cong \operatorname{Ker}(M \otimes^{\mu(M)} M \to M) \cong \Delta_1 V(M, \mu(M), \mu) \cong \Delta_1 V(M, P, \mu)$. □

Example 1

(1) It was shown in Proposition 2, that the Conduché and Ellis homology group $H_2(M)_P$ is a quotient of $\Delta_1 V(M,P,\mu)$. We will now see that in general these two groups are non-isomorphic.
Consider the crossed module $(A,1,1)$ where A is an abelian group. Then $\Delta_1 V(A,1,1) \cong A \otimes^1 A \cong A \otimes_{\mathbb{Z}} A$, while $H_2(A)_1 \cong A \wedge_1 A \cong A \wedge A$ which is isomorphic to the integral homology $H_2(A)$ of the group A. In general these groups are different; for example $\Delta_1 V(\mathbb{Z},1,1) \cong \mathbb{Z}$ but $H_2(\mathbb{Z})_1 = 0$.
(2) On the other hand, by Corollary 2, $\Delta_1 V(F,F,\mathrm{Id})$ and $H_2(F)_F$ coincide if F is a free group. For example $H_2(\mathbb{Z})_{\mathbb{Z}} \cong \Delta_1 V(\mathbb{Z},\mathbb{Z},\mathrm{Id}) \cong \mathbb{Z} \otimes_{\mathbb{Z}} \mathbb{Z} \cong \mathbb{Z}$.

Remark 7 In [10], Casas and Van der Linden study the existence and construction of universal central extensions relative to a Birkhoff subcategory in the context of semi-abelian categories. When universal central extensions of precrossed modules with respect to $\mathcal{CM}$ are considered they prove the existence of such extensions for precrossed modules (M,P,μ) which are *perfect*, in the sense that M is a perfect group and $P = 1$.

In this context, an extension of precrossed modules $(N,Q,\delta) \rightarrowtail (Y,X,\delta) \overset{p}{\twoheadrightarrow} (M,P,\mu)$ is central if and only if $N \subset \{y \in Y \mid \langle y,y'\rangle = 1 = \langle y',y\rangle \text{ for every } y' \in Y\}$.

For a perfect precrossed module $(M,1,1)$, Casas and Van der Linden [10] prove that its universal central extension is $\Delta V(M,1,1) \rightarrowtail DV(M,1,1) \overset{p}{\twoheadrightarrow} (M,1,1)$. This extension is trivial in the second component and has as first component the group extension $\Delta_1 V(M,1,1) \rightarrowtail M \otimes^1 M \overset{\kappa_M}{\twoheadrightarrow} M$, which coincides with the classical universal central extension of the perfect group M [27], since by Corollary 2 $\Delta_1 V(M,1,1) \cong H_2(M)_1 \cong H_2(M)$, and by Lemma 1 $M \otimes^1 M \cong M \wedge_1 M$ which for a perfect group M coincides with the non-abelian tensor product of groups $M \otimes M$ defined by Brown and Loday in [8].

5 Higher dimensional Baer invariants

In [30], Modi defines higher dimensional Baer invariants in a variety of Ω-groups relative to a subvariety, using Keune's homotopical theory [28].

The nth *Baer invariant* $\mathcal{B}_n(M,P,\mu)$ of a precrossed module (M,P,μ) relative to the variety of crossed modules is defined as the value in (M,P,μ) of the nth left derived functor of the Peiffer abelianization

$$\mathcal{B}_n(M,P,\mu) = L_n U(M,P,\mu) = \Pi_n\big(U(W.,F.,\tau.)\big), \quad n \geqslant 1\,,$$

where $(W.,F.,\tau.)$ is a free simplicial resolution of (M,P,μ) and $\Pi_n\big(U(W.,F.,\tau.)\big)$ denotes the nth homology group of $N\big(U(W.,F.,\tau.)\big)$, the *normalized Moore chain complex* associated to $U(W.,F.,\tau.)$.

The following result taken from [19], establishes that the first Baer invariant $\mathcal{B}_1$ is naturally isomorphic to the Baer invariant ΔV:

Theorem 4 *[19, Theorem 6.9] For every precrossed module* (M, P, μ) *there is an isomorphism* $\mathcal{B}_1(M, P, \mu) \cong \Delta V(M, P, \mu)$.

Remark 8 Theorem 4 says that $\mathcal{B}_1(M, P, \mu) \cong (\Delta_1 V(M, P, \mu), 1, 1)$. Actually, the second component of the simplicial crossed module $U(W., F., \tau.)$ is the same as the second component of the free simplicial resolution $(W., F., \tau.)$ of (M, P, μ). Thus the second components of the higher dimensional Baer invariants $\mathcal{B}_n(M, P, \mu)$ are zero.

We will denote by $\Delta_n V(M, P, \mu)$ the first component of $\mathcal{B}_n(M, P, \mu)$, so in general $\mathcal{B}_n(M, P, \mu) = (\Delta_n V(M, P, \mu), 1, 1)$.

The higher dimensional Baer invariants $\mathcal{B}_n(M, P, \mu)$ of precrossed modules relative to the variety of crossed modules play a fundamental role in the study of the relationship between the cohomology theories defined in the varieties of Ω-groups $\mathcal{PCM}$ and $\mathcal{CM}$.

In [2] cohomology groups of a precrossed module with trivial coefficients were defined. We can extend this definition to a cohomology theory with arbitrary coefficients in accordance with the standard definition of cohomology for varieties of Ω-groups [11]. Thus the nth *cohomology group* $H^n\big((M, P, \mu), (A, B, f)\big)$ of a precrossed module (M, P, μ) with coefficients in a (M, P, μ)-module (A, B, f) is defined as the $(n-1)$th cohomology group of the cochain complex of abelian groups

$$\mathrm{Der}\,\big((W., F., \tau.), (A, B, f)\big)$$

where the group of *derivations* $\mathrm{Der}\,\big((W_n, F_n, \tau_n), (A, B, f)\big)$ of the precrossed module (W_n, F_n, τ_n) into the (W_n, F_n, τ_n)-module (A, B, f) is defined as the homomorphism group

$$\mathrm{Hom}_{\mathcal{PCM}/(M,P,\mu)} \begin{pmatrix} (W_n, F_n, \tau_n) & & (A \rtimes M, B \rtimes P, \{f, \mu\}) \\ \downarrow & , & \downarrow \\ (M, P, \mu) & & (M, P, \mu) \end{pmatrix}$$

Remark 9 The nth *cohomology group* $H^n\big((M, P, \mu), (A, B, f)\big)$ of a precrossed module (M, P, μ) with coefficients in an abelian precrossed module (A, B, f) was introduced in [2] as the $(n-1)$th cohomology group of the cochain complex of abelian groups $\mathrm{Hom}_{\mathcal{PCM}}\,\big((W., F., \tau.), (A, B, f)\big)$, which coincides with the cohomology defined above in the particular case that (A, B, f) is considered as the trivial (M, P, μ)-module

$$(A, B, f) \rightarrowtail (A \times M, B \times P, \{f, \mu\}) \rightleftarrows (M, P, \mu)$$

where $B \times P$ acts on $A \times M$ componentwise.

Cohomology groups $H^n_{\mathcal{CM}}\big((T, G, \partial), (A, B, f)\big)$ of a crossed module (T, G, ∂) with coefficients in a (T, G, ∂)-module (A, B, f) in the category $\mathcal{CM}$ were introduced by a similar procedure in [9] and [31], by using free simplicial resolutions of crossed modules, as an instance of Barr and Beck's general theory [3].

Remark 10 The Baer invariants $\mathcal{B}_n(M, P, \mu)$ are abelian precrossed modules [19, Theorem 5.5]. Actually they are in a natural way $U(M, P, \mu)$-modules in $\mathcal{CM}$ ([11]) and therefore (M, P, μ)-modules in $\mathcal{PCM}$.

The following theorem establishes the connection between the generalized Baer invariants and the cohomology invariants mentioned in Remark 9:

Theorem 5 *For a precrossed module (M, P, μ) and an $U(M, P, \mu)$-module (A, B, f) in $\mathcal{CM}$, there exists an exact sequence of abelian groups*

$$\begin{aligned}
0 &\longrightarrow H^2_{\mathcal{CM}}\Big(\Big(\tfrac{M}{\langle M,M\rangle}, P, \overline{\mu}\Big), (A, B, f)\Big) \longrightarrow H^2\big((M, P, \mu), (A, B, f)\big) \\
&\longrightarrow \mathrm{Hom}_{M \rtimes P}\left(\Delta_1 V(M, P, \mu), \mathrm{Ker}(f)\right) \\
&\longrightarrow H^3_{\mathcal{CM}}\Big(\Big(\tfrac{M}{\langle M,M\rangle}, P, \overline{\mu}\Big), (A, B, f)\Big) \longrightarrow H^3\big((M, P, \mu), (A, B, f)\big).
\end{aligned}$$

Moreover, if $\Delta_i V(M, P, \mu) = 0$ for $1 \leqslant i \leqslant n-1$, there also exists an exact sequence of abelian groups

$$\begin{aligned}
0 &\longrightarrow H^{n+1}_{\mathcal{CM}}\Big(\Big(\tfrac{M}{\langle M,M\rangle}, P, \overline{\mu}\Big), (A, B, f)\Big) \longrightarrow H^{n+1}\big((M, P, \mu), (A, B, f)\big) \\
&\longrightarrow \mathrm{Hom}_{M \rtimes P}\left(\Delta_n V(M, P, \mu), \mathrm{Ker}(f)\right) \\
&\longrightarrow H^{n+2}_{\mathcal{CM}}\Big(\Big(\tfrac{M}{\langle M,M\rangle}, P, \overline{\mu}\Big), (A, B, f)\Big) \longrightarrow H^{n+2}\big((M, P, \mu), (A, B, f)\big).
\end{aligned}$$

Proof Applying [11, Theorem 9] we obtain a five term exact sequence

$$\begin{aligned}
0 &\longrightarrow H^2_{\mathcal{CM}}\Big(\Big(\tfrac{M}{\langle M,M\rangle}, P, \overline{\mu}\Big), (A, B, f)\Big) \longrightarrow H^2\big((M, P, \mu), (A, B, f)\big) \\
&\longrightarrow \mathrm{Hom}_{(M,P,\mu)}\left(\mathcal{B}_1(M, P, \mu), (A, B, f)\right) \\
&\longrightarrow H^3_{\mathcal{CM}}\Big(\Big(\tfrac{M}{\langle M,M\rangle}, P, \overline{\mu}\Big), (A, B, f)\Big) \longrightarrow H^3\big((M, P, \mu), (A, B, f)\big)
\end{aligned}$$

where $\mathrm{Hom}_{(M,P,\mu)}\big(\mathcal{B}_1(M,P,\mu),(A,B,f)\big)$ denotes the group of (M,P,μ)-module morphisms from $\mathcal{B}_1(M,P,\mu)$ to (A,B,f), that is, the group of precrossed module morphisms $(h,1)\colon (\Delta_1 V(M,P,\mu),1,1) \to (A,B,f)$ such that the following diagram commutes

$$\begin{tikzcd}
(\Delta_1 V(M,P,\mu),1,1) \arrow[r, tail] \arrow[d, "{(h,1)}"'] & (\Delta_1 V(M,P,\mu) \rtimes M, 1 \rtimes P, \{1,\mu\}) \arrow[r, shift left, two heads] \arrow[d] & (M,P,\mu) \arrow[l, shift left, tail] \arrow[d, equal] \\
(A,B,f) \arrow[r, tail] & (A \rtimes M, B \rtimes P, \{f,\mu\}) \arrow[r, shift left, two heads] & (M,P,\mu) \arrow[l, shift left, tail]
\end{tikzcd}$$

or equivalently the group of morphisms $h \times 1\colon \Delta_1 V(M,P,\mu) \times 1 \to A \times B$ of Ω-groups which are compatible with the $(M \rtimes P)$-actions.

Remark that $(h,1)\colon (\Delta_1 V(M,P,\mu),1,1) \to (A,B,f)$ is a precrossed module morphism if and only if $\mathrm{Im}(h) \subset \mathrm{Ker}(f)$, and that $\mathrm{Ker}(f)$ is an $M \rtimes P$-subgroup of $A \times B$. Then $\mathrm{Hom}_{(M,P,\mu)}\big(\mathcal{B}_1(M,P,\mu),(A,B,f)\big)$ is isomorphic to the group $\mathrm{Hom}_{M\rtimes P}\big(\Delta_1 V(M,P,\mu),\mathrm{Ker}(f)\big)$ of group homomorphisms which preserve the $(M \rtimes P)$-action.

The other sequence can be obtained analogously from [11, Theorem 9]. □

The following Corollary is a particular case of [11, Corollary 10].

Corollary 3 *For a precrossed module (M,P,μ) and a fixed $n \geqslant 1$, the following conditions are equivalent:*

(i) $\mathcal{B}_i(M,P,\mu) = 0$ *for* $1 \leqslant i \leqslant n$*;*

(ii) $H^i_{\mathcal{CM}}\Big(\big(\frac{M}{\langle M,M\rangle},P,\overline{\mu}\big),(A,B,f)\Big) \cong H^i\big((M,P,\mu),(A,B,f)\big)$ *for* $2 \leqslant i \leqslant n+1$ *and the morphism*

$$H^{n+2}_{\mathcal{CM}}\left(\Big(\frac{M}{\langle M,M\rangle},P,\overline{\mu}\Big),(A,B,f)\right) \to H^{n+2}\big((M,P,\mu),(A,B,f)\big)$$

is a monomorphism for each $U(M,P,\mu)$-module (A,B,f) in $\mathcal{CM}$.

Remark 11 It was shown in [1] and [9] that the groups $H^2\big((M,P,\mu),(A,B,f)\big)$ and $H^2_{\mathcal{CM}}\Big(\big(\frac{M}{\langle M,M\rangle},P,\overline{\mu}\big),(A,B,f)\Big)$ classify central extensions of precrossed and crossed modules, respectively, whenever we take as coefficients of the cohomologies an arbitrary abelian precrossed module (A,B,f) with trivial action. These results are special cases of a general result for the second cohomology group in semi-abelian categories [24, Theorem 6.3]. The first five term exact sequence in Theorem 5 connects the quotient group $\mathrm{Cext}\big((M,P,\mu),(A,B,f)\big)$ of congruence classes of central extensions of (M,P,μ) by (A,B,f) in $\mathcal{PCM}$ with the group $\mathrm{Cext}_{\mathcal{CM}}\Big(\big(\frac{M}{\langle M,M\rangle},P,\overline{\mu}\big),(A,B,f)\Big)$ of congruence classes of central extensions of its Peiffer abelianization by (A,B,f) in $\mathcal{CM}$.

For example, if (M,P,μ) is a precrossed module such that $\Delta_1 V(M,P,\mu) = 0$ or if f is an injective group homomorphism, we have a group isomorphism:

$$\mathrm{Cext}\big((M,P,\mu),(A,B,f)\big) \cong \mathrm{Cext}_{\mathcal{CM}}\left(\Big(\frac{M}{\langle M,M\rangle},P,\overline{\mu}\Big),(A,B,f)\right).$$

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