Central Extensions of Precrossed Modules *

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Abstract. We classify the precrossed module central extensions using the second cohomology group of precrossed modules. We relate these central extensions to the relative central group extensions of Loday, and to other notions of centrality defined in general contexts. Finally we establish a Universal Coefficient Theorem for the (co)homology of precrossed modules, which we use to describe the precrossed module central extensions in terms of the generalized Galois theory developed by Janelidze.

Keywords: central extension, precrossed module, cohomology, Galois theory.

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1. Introduction

In this paper we extend the description of the central extensions of crossed modules developed in [5], to the category of precrossed modules, using the cohomological tools defined by the authors in [1].

In [5] it is proved that the second cohomology group of a crossed module (T, G, ∂) with coefficients in an abelian crossed module (A, B, f) defined in that paper, which we denote by

$$H^2_{CCG}\left((T, G, \partial), (A, B, f)\right)$$

classifies all the crossed module central extensions of (T, G, ∂) by (A, B, f), and so it generalizes the classical result of classification of group extensions with trivial actions.

We will go further on. We will show that the second cohomology group of a precrossed module (M, P, μ) with coefficients in an abelian

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precrossed module (A, B, f), defined in [1]

$$H^2((M, P, \mu), (A, B, f))$$

classifies the precrossed module central extensions of (M, P, μ) by (A, B, f). When (M, P, μ) and (A, B, f) are crossed modules, the resulting set of congruence classes of precrossed module central extensions $Cext_{\mathcal{PCM}}((M, P, \mu), (A, B, f))$ contains the set of congruence classes of crossed module central extensions $Cext_{\mathcal{CM}}((M, P, \mu), (A, B, f))$ studied in [5].

We will also prove that in certain particular cases, the congruence classes of precrossed module central extensions coincide with the congruence classes of *relative central extensions* of groups studied by Loday [17].

We begin Section 1 by recalling the definition of the cohomology of precrossed modules, as it was stated in [1].

In Section 2 we introduce the notion of central extension of a precrossed module, and we relate this definition to other notions of centrality defined in some general contexts, as in exact categories [13], or in varieties of Ω-groups ([8] and [18]).

In Section 3 we establish our result of classification of precrossed module central extensions.

Finally, in Section 4, we prove a Universal Coefficient Theorem for the (co)homology of precrossed modules, which relates the cohomology with the homology of precrossed modules, through certain exact sequences. We apply this Universal Coefficient Theorem to describe the central extensions in $\mathcal{PCM}$ in terms of the generalized Galois theory developed by Janelidze in [10], [11] and [12].

2. Cohomology of precrossed modules

We begin this section by recalling certain basic aspects from the theory of precrossed modules. Details can be found in [1].

A *precrossed module* (M, P, μ) is a group homomorphism $\mu : M \to P$ together with an action of P on M, denoted by ${}^{p}m$ for $p \in P$ and $m \in M$, and satisfying $\mu({}^{p}m) = p\mu(m)p^{-1}$ for all $p \in P$ and $m \in M$. If in addition it satisfies the Peiffer's identity ${}^{\mu(m)}m' = mm'm^{-1}$ for all $m, m' \in M$, we say that (M, P, μ) is a *crossed module*.

A *precrossed module morphism* $(\Phi, \Psi) : (M_1, P_1, \mu_1) \to (M_2, P_2, \mu_2)$ is a pair of group homomorphisms $\Phi : M_1 \to M_2$ and $\Psi : P_1 \to P_2$ such that $\Psi \circ \mu_1 = \mu_2 \circ \Phi$ and $\Phi({}^{p}m) = {}^{\Psi(p)}\Phi(m)$ for all $p \in P_1$ and $m \in M_1$.

We denote the category of precrossed (crossed) modules by $\mathcal{PCM}$ ($\mathcal{CM}$). Janelidze and Pedicchio enumerate in [14] a list of "internal notions" in a variety of universal algebras. It is known that the notions of precrossed module and crossed module are respectively equivalent to the notions of *internal reflexive graph* and *internal category* in the category of groups. The internal reflexive graph equivalent to a precrossed module (M, P, μ) is the graph with P as the group of objects and with the semidirect product $M \rtimes P$ as the group of morphisms, and with

$$\begin{array}{rl} s : M \rtimes P \longrightarrow P & \qquad b : M \rtimes P \longrightarrow P \\ (m, p) \longmapsto p & \qquad (m, p) \longmapsto \mu(m)p \end{array}$$

Next we recall notions like injection, surjection, (normal) subobject, image, abelian object, commutator, centre, etc..., which are not new, but come up as the appropriate instances of known categorical notions.

A morphism (Φ, Ψ) in $\mathcal{PCM}$ is said to be *injective (surjective)* if both Φ and Ψ are injective (surjective) group homomorphisms.

A *precrossed submodule* (N, Q, μ') of a precrossed module (M, P, μ) is a precrossed module such that N and Q are, respectively, subgroups of M and P, the action of Q on N is induced by the one of P on M and $\mu_{|N} = \mu'$. It is said to be a *normal precrossed submodule* if besides N and Q are normal in M and P, ${}^{p}n \in N$ and ${}^{q}mm^{-1} \in N$ for all $p \in P$, $q \in Q$, $m \in M$ and $n \in N$.

If (N, Q, μ) is a normal precrossed submodule of (M, P, μ), we define the *quotient precrossed module* $(M, P, \mu)/(N, Q, \mu)$ as $(M/N, P/Q, \overline{\mu})$ where the homomorphism $\overline{\mu}$ is induced by μ and P/Q acts on M/N by ${}^{pQ}mN = ({}^{p}m)N$ for $p \in P$ and $m \in M$.

We call *Peiffer subgroup* $\langle M, M\rangle$ of a precrossed module (M, P, μ) the subgroup of M generated by the Peiffer elements $m_1 m_2 m_1^{-1} {}^{\mu(m_1)}m_2^{-1}$ with $m_1, m_2 \in M$. It is a normal subgroup of M, and the quotient $(M, P, \mu)_{cr} = (M, P, \mu)/(\langle M, M\rangle, 1, 1) = (M/\langle M, M\rangle, P, \overline{\mu})$ is a crossed module.

The *kernel* of a precrossed module morphism $(\Phi, \Psi) : (M_1, P_1, \mu_1) \to (M_2, P_2, \mu_2)$ is the normal precrossed submodule $(Ker\Phi, Ker\Psi, \mu_1)$ of (M_1, P_1, μ_1). Its *image* is the precrossed submodule $(Im\Phi, Im\Psi, \mu_2)$ of (M_2, P_2, μ_2).

In [1] we introduced analogues to some basic concepts from group theory, like centre or commutator groups, in the category of precrossed modules. In the case of crossed modules these concepts were introduced by Norrie [20].

The *centre* $Z(M, P, \mu)$ of a precrossed module (M, P, μ) is the normal precrossed submodule $(Inv(M) \cap Z(M), St_P(M) \cap Z(P), \mu)$, where $St_P(M)$ denotes the group $\{p \in P \mid {}^{p} m = m \text{ for all } m \in M\}$, $Inv(M) = \{m \in M \mid \mu(m) \in St_P(M) \text{ and } {}^{p}m = m \text{ for all } p \in P\}$ and $Z(M)$, $Z(P)$

denote the centres of M and P. $Z(M,P,\mu)$ is the maximal central precrossed submodule of (M,P,μ).

A precrossed module (M,P,μ) is said to be *abelian* if $(M,P,\mu) = Z(M,P,\mu)$. Equivalently M and P are abelian groups and P acts trivially on M.

If (N,Q,μ) and (R,K,μ) are normal precrossed submodules of (M,P,μ), we define the *commutator precrossed submodule* $[(N,Q,\mu),(R,K,\mu)]$ *of* (N,Q,μ) *and* (R,K,μ) as the normal precrossed submodule $([Q,R]\,[K,N]\,[N,R]\,,\ [Q,K]\,,\mu)$ of (M,P,μ), where $[Q,R]$ denotes the normal subgroup of M generated by the elements $\left\{{}^{q}rr^{-1} \mid q \in Q, r \in R\right\}$, $[K,N]$ denotes the normal subgroup of M generated by the elements $\left\{{}^{k}nn^{-1} \mid k \in K, n \in N\right\}$ and $[N,R]$ and $[Q,K]$ denote the usual commutator subgroups of N with R and Q with K.

In particular, the *commutator precrossed submodule* of a precrossed module (M,P,μ) is $[(M,P,\mu),(M,P,\mu)] = ([M,M]\,[P,M],\ [P,P]\,,\mu)$. It is the smallest normal precrossed submodule of (M,P,μ) making the quotient an abelian precrossed module.

The inclusion of abelian precrossed modules $\mathcal{APCM}$ in $\mathcal{PCM}$ has a left adjoint $ab : \mathcal{PCM} \to \mathcal{APCM}$ termed the *abelianisation functor*, which assigns to a precrossed module (M,P,μ) the abelian precrossed module $(M,P,\mu)_{ab} = (M/\ [M,M]\,[P,M]\,,\ P/\,[P,P]\,,\overline{\mu})$.

The forgetful functor $\mathcal{U} : \mathcal{PCM} \to \mathcal{S}et$, $\mathcal{U}(M,P,\mu) = M \times P$, that assigns to each precrossed module (M,P,μ) the cartesian product of the underlying sets M and P, is tripleable. Its left adjoint, the free precrossed module functor $\mathcal{F} : \mathcal{S}et \to \mathcal{PCM}$ is given by $\mathcal{F}(X) = (\overline{F}, F*F, \langle i_1, Id\rangle_{|\overline{F}})$, where F is the free group over X, $\overline{F} = Ker(F*(F*F) \xrightarrow{\langle 0,Id\rangle} F*F)$, $\langle i_1, Id\rangle : F*(F*F) \longrightarrow F*F$, $i_1 : F \rightarrowtail F*F$ is the first inclusion in the coproduct, and $F*F$ acts on $\overline{F}$ by conjugation. Note that the forgetful functor $\mathcal{U}$ is nothing but the underlying set of the group of morphisms of the corresponding internal reflexive graph.

The category $\mathcal{PCM}$ has enough projective objects, since it is equivalent with the category of internal reflexive graphs. Each precrossed module admits a presentation as a quotient of a projective precrossed module by means of the counit of the adjunction between $\mathcal{U}$ and $\mathcal{F}$. A useful construction of a family of projective precrossed modules can be found in [1].

For a precrossed module (M,P,μ) let us consider the cotriple resolution $C_*(M,P,\mu) \to (M,P,\mu)$ associated to the functors $\mathcal{F}$ and $\mathcal{U}$. Specializing Barr and Beck's cotriple cohomology [3], we define as in [1], for $n \geq 1$, the *homology precrossed modules* of the precrossed module (M,P,μ) by

$$H_n(M,P,\mu) = H_{n-1}\left((C_*(M,P,\mu))_{ab}\,,\partial_*\right)$$

and the *cohomology groups* of the precrossed module (M,P,μ) with coefficients in an abelian precrossed module (A,B,f) by

$$H^n((M,P,\mu),(A,B,f)) = H^{n-1}(Hom_{\mathcal{PCM}}(C_*(M,P,\mu),(A,B,f)),\partial^*)$$

THEOREM 1. *(Five term exact sequence for the cohomology of precrossed modules)*

Let $0 \to (N,Q,\mu) \overset{i}{\rightarrowtail} (M,P,\mu) \overset{k}{\twoheadrightarrow} (L,C,\omega) \to 0$ *be an exact sequence of precrossed modules. For each abelian precrossed module* (A,B,f)*, there exists an exact sequence of abelian groups*

$$\begin{aligned} 0 &\longrightarrow H^1((L,C,\omega),(A,B,f)) \longrightarrow H^1((M,P,\mu),(A,B,f)) \longrightarrow \\ &\longrightarrow Hom_{\mathcal{PCM}}\left(\frac{(N,Q,\mu)}{[(M,P,\mu),(N,Q,\mu)]},(A,B,f)\right) \longrightarrow \\ &\longrightarrow H^2((L,C,\omega),(A,B,f)) \longrightarrow H^2((M,P,\mu),(A,B,f)) \end{aligned}$$

The proof of Theorem 1 is analogous to the one of [5, Theorem 12], with the help of [2, Lemma 3].

To establish our result of classification of central extensions we will need the following result, which is the counterpart for the cohomology of the Hopf's formula for the homology of precrossed modules [2, Corollary 4].

COROLLARY 2. *If* $(V,R,\tau) \rightarrowtail (W,F,\tau) \overset{\pi}{\twoheadrightarrow} (M,P,\mu)$ *is a projective presentation of a precrossed module* (M,P,μ)*, and* (A,B,f) *is an abelian precrossed module, then there exists a natural isomorphism of abelian groups between* $H^2((M,P,\mu),(A,B,f))$ *and the cokernel of the group homomorphism*

$$Hom_{\mathcal{PCM}}((W,F,\tau),(A,B,f)) \to Hom_{\mathcal{PCM}}\left(\frac{(V,R,\tau)}{[(W,F,\tau),(V,R,\tau)]},(A,B,f)\right)$$

Proof.

Apply the five term exact sequence in cohomology to the projective presentation of (M,P,μ). The result follows since $H^2((W,F,\tau),(A,B,f)) = 0$.

REMARK 3.

The analogous classical theorems for the cohomology of groups with trivial coefficients can be deduced from these results. If we take an extension of groups $N \rightarrowtail G \twoheadrightarrow Q$ *and an abelian group* A*, we can consider them as a sequence of precrossed modules* $(1,N,i) \rightarrowtail (1,G,i) \twoheadrightarrow (1,Q,i)$ *and an abelian precrossed module* $(1,A,i)$*, and applying [1, Theorem 4.1] we get that the resulting five term exact sequence of Theorem 1 is the five term exact sequence for the cohomology of groups*

$$0 \to H^1(Q,A) \to H^1(G,A) \to Hom_{\mathcal{G}\mathrm{r}}\left(\frac{N}{[G,N]},A\right) \to H^2(Q,A) \to H^2(G,A)$$

On the other hand, a classical result of Eilenberg and MacLane [6, Theorem 3.1] can be obtained from Corollary 2. For a free presentation $R \rightarrowtail F \overset{\pi}{\twoheadrightarrow} G$ *of a group* G*, there is a projective presentation* $(1, R, j) \rightarrowtail (1, F, j) \overset{\pi}{\twoheadrightarrow} (1, G, i)$ *of the precrossed module* $(1, G, i)$*. Applying [1, Theorem 4.1] and Corollary 2, and taking as coefficients the abelian precrossed module* $(1, A, i)$*, we obtain*

$$H^2(G, A) \cong Coker\left(Hom_{\mathcal{G}\mathrm{r}}(F, A) \longrightarrow Hom_{\mathcal{G}\mathrm{r}}\left(\frac{R}{[F, R]}, A\right)\right)$$

3. Definition of central extension. Equivalence with the notion of Janelidze-Kelly.

Our notion of central extension for precrossed modules can be defined by means of the definition of centre of a precrossed module:

DEFINITION 4. *Let* (M, P, μ) *be a precrossed module, and* (A, B, f) *an abelian precrossed module. A central extension of* (M, P, μ) *by* (A, B, f) *is an extension*

$$E : (A, B, f) \rightarrowtail (Y, X, \delta) \twoheadrightarrow (M, P, \mu)$$

such that $(A, B, f) \subset Z(Y, X, \delta)$*.*

In [13], a notion of central extension in an exact category $\mathcal{C}$, relative to an "admissible" subcategory $\mathcal{X}$ of $\mathcal{C}$, is introduced. When $\mathcal{C}$ is a Mal'tsev category, every Birkhoff subcategory of $\mathcal{C}$ is admissible.

We can consider the categorical theory of central extensions in $\mathcal{PCM}$, since the category $\mathcal{PCM}$ is equivalent to a variety of Ω-groups [16] (concretely to the variety of groups with operators $\{s, b\}$ satisfying the relations $bs = s$ and $sb = b$), and so $\mathcal{PCM}$ is a Barr exact Mal'tsev category. Our definition of central extension is equivalent to the categorical one applied to the category $\mathcal{PCM}$ of precrossed modules with the admissible subcategory $\mathcal{APCM}$ of abelian precrossed modules.

Let us explain this in detail. Consider the adjunction $ab \dashv \mathcal{J}$,

$$\begin{array}{ccc} \mathcal{PCM} & (M, P, \mu) & (A, B, f) \\ ab \downarrow\uparrow \mathcal{J} & ab \downarrow & \uparrow \mathcal{J} \\ \mathcal{APCM} & (M, P, \mu)_{ab} & (A, B, f) \end{array}$$

where $\mathcal{J} : \mathcal{APCM} \to \mathcal{PCM}$ is the inclusion of the Birkhoff variety $\mathcal{APCM}$ in $\mathcal{PCM}$, and $ab : \mathcal{PCM} \to \mathcal{APCM}$ is the abelianisation functor. Following [13], an extension $f : A \twoheadrightarrow B$ is called *trivial* if the following diagram is a pullback

$$\begin{array}{ccc} A & \longrightarrow & (A)_{ab} \\ f\downarrow & & \downarrow (f)_{ab} \\ B & \longrightarrow & (B)_{ab} \end{array}$$

where the horizontal morphisms are given by the unit of the adjunction. Janelidze and Kelly call an extension $f : A \twoheadrightarrow B$ a *central extension* [13, pag. 152], if there exists an extension $p : E \twoheadrightarrow B$ of B such that in the pullback

$$\begin{array}{ccc} E \underset{B}{\times} A & \overset{\pi_2}{\to} & A \\ \pi_1\downarrow & & \downarrow f \\ E & \underset{p}{\to} & B \end{array}$$

the morphism π_1 is a trivial extension.

PROPOSITION 5. *Our notion of centrality coincides with the one of Janelidze and Kelly in the case that we take the category $\mathcal{PCM}$ and the "admissible" subcategory $\mathcal{APCM}$.*

Proof.

Given an extension of precrossed modules

$$(A, B, f) \rightarrowtail (Y, X, \delta) \twoheadrightarrow (M, P, \mu)$$

such that $(A, B, f) \subset Z(Y, X, \delta)$, it is verified that in the pullback

$$\begin{array}{ccc} \left(Y \underset{M}{\times} Y, X \underset{P}{\times} X, \delta \times \delta\right) & \overset{\pi_2}{\to} & (Y, X, \delta) \\ \pi_1\downarrow & & \downarrow \\ (Y, X, \delta) & \to & (M, P, \mu) \end{array}$$

the extension $\left(Y \underset{M}{\times} Y, X \underset{P}{\times} X, \delta \times \delta\right) \overset{\pi_1}{\twoheadrightarrow} (Y, X, \delta)$ is trivial, that is, the diagram

$$\begin{array}{ccc} \left(Y \underset{M}{\times} Y, X \underset{P}{\times} X, \delta \times \delta\right) & \to & \left(Y \underset{M}{\times} Y, X \underset{P}{\times} X, \delta \times \delta\right)_{ab} \\ \pi_1\downarrow & & \downarrow (\pi_1)_{ab} \\ (Y, X, \delta) & \overset{\eta}{\to} & (Y, X, \delta)_{ab} \end{array}$$

is a pullback. To prove it, we will construct an isomorphism $\lambda = (\lambda_1, \lambda_2)$ between

$$\left(Y \underset{M}{\times} Y, X \underset{P}{\times} X, \delta \times \delta\right)$$

and the fiber product

$$\left(Y \underset{\frac{Y}{[Y,Y][X,Y]}}{\times} \left(\frac{Y \underset{M}{\times} Y}{\left[Y \underset{M}{\times} Y, Y \underset{M}{\times} Y\right]\left[X \underset{P}{\times} X, Y \underset{M}{\times} Y\right]}\right), X \underset{X_{ab}}{\times} \left(X \underset{P}{\times} X\right)_{ab}, \delta \times (\delta \times \delta)_{ab}\right)$$

of $(\pi_1)_{ab}$ with η, defined by

$$\lambda_1 : Y \underset{M}{\times} Y \longrightarrow Y \underset{\frac{Y}{[Y,Y][X,Y]}}{\times} \left(\frac{Y \underset{M}{\times} Y}{\left[Y \underset{M}{\times} Y, Y \underset{M}{\times} Y\right]\left[X \underset{P}{\times} X, Y \underset{M}{\times} Y\right]}\right)$$
$$(y, z) \rightsquigarrow \left(y, \overline{(y, z)}\right)$$
$$\lambda_2 : X \underset{P}{\times} X \longrightarrow X \underset{X_{ab}}{\times} \left(X \underset{P}{\times} X\right)_{ab}$$
$$(t, x) \rightsquigarrow \left(t, \overline{(t, x)}\right)$$

On the other hand, given an extension of precrossed modules

$$(A, B, f) \rightarrowtail (Y, X, \delta) \overset{(\varphi_1,\varphi_2)}{\twoheadrightarrow} (M, P, \mu)$$

which is central in the sense of Janelidze and Kelly, we will prove that $(A, B, f) \subset Z(Y, X, \delta)$. Let $(S, H, \gamma) \overset{(\psi_1,\psi_2)}{\twoheadrightarrow} (M, P, \mu)$ be the extension for which the diagram

$$\begin{array}{ccc} \left(S \underset{M}{\times} Y, H \underset{P}{\times} X, \gamma \times \delta\right) & \rightarrow & \left(S \underset{M}{\times} Y, H \underset{P}{\times} X, \gamma \times \delta\right)_{ab} \\ \pi_1 \downarrow & & \downarrow (\pi_1)_{ab} \\ (S, H, \gamma) & \rightarrow & (S, H, \gamma)_{ab} \end{array}$$

is a pullback. The kernel of π_1 is given by the injective morphism $(A, B, f) \rightarrowtail \left(S \underset{M}{\times} Y, H \underset{P}{\times} X, \gamma \times \delta\right)$, and the kernel of $(\pi_1)_{ab}$ is then given by another injective morphism $(\theta, \kappa) : (A, B, f) \rightarrowtail (S \underset{M}{\times} Y, H \underset{P}{\times} X,$ $\gamma \times \delta)_{ab}$ defined by $\theta(a) = \overline{(0, a)}$ and $\kappa(b) = \overline{(0, b)}$. Now, it is straightforward to verify that $[(Y, X, \delta), (A, B, f)] = 0$. For example, we will prove that $[X, B] = 0$: a generator $xbx^{-1}b^{-1}$ equals zero, with $x \in X$

and $b \in B$, if and only if $\kappa(xbx^{-1}b^{-1}) = \overline{(0, xbx^{-1}b^{-1})}$ is zero. Taking $h \in H$ such that $\psi_2(h) = \varphi_2(x)$, we get that $\overline{(0, xbx^{-1}b^{-1})} = \overline{(0,x)(h,b)(0,x)}^{-1}\overline{(h,b)}^{-1} = \overline{(0,x)(0,x)}^{-1}\overline{(h,b)(h,b)}^{-1} = 0$.

Janelidze and Kelly show in [13] that their categorical notion of central extension, generalizes the notion of centrality developed by Fröhlich [8] and Lue [18] for a pair composed by a variety of $\Omega-$groups $\mathcal{C}$, and a subvariety $\mathcal{X}$. On the other hand, the notion of centrality of Fröhlich is, a generalization of the classical theory of central group extensions, in case we consider the variety of groups $\mathcal{G}r$ and the subvariety of abelian groups $\mathcal{A}b$.

Also, Janelidze and Kelly, showed in [15] that the categorical notion of centrality generalizes the notion of centrality in universal algebra, defined through the theory of commutators, for each congruence modular variety $\mathcal{C}$ and subvariety $\mathcal{X}$ (see for example [7]).

The pair formed by the category $\mathcal{PCM}$ and the subcategory $\mathcal{APCM}$ satisfy the conditions required by each of the different mentioned contexts. So, by Proposition 5, our notion of central extension is a special case of any of them. Using the fact that the category $\mathcal{PCM}$ is semi-abelian, some other characterizations of our central extensions can also be found. In [4], Bourn and Gran search for new characterizations of the central extensions in a semi-abelian category $\mathcal{C}$, when as "admissible" subcategory is taken the category of the abelian group objects $\mathcal{A}b\mathcal{C}$ in $\mathcal{C}$. For example, they obtain the following result: an extension $f : A \twoheadrightarrow B$ in $\mathcal{C}$ is central with respect to $\mathcal{A}b\mathcal{C}$ if and only if the subdiagonal morphism s_0 is a normal monomorphism,

$$R\,[f] \underset{}{\overset{\overset{s_0}{\curvearrowleft}}{\rightrightarrows}} A \xrightarrow{f} B$$

where $R\,[f] \rightrightarrows A$ is the kernel pair of f.

4. Classification of central extensions of precrossed modules

DEFINITION 6. *Two extensions E_1 and E_2 of (M, P, μ) by (A, B, f) are congruent if and only if there exists a morphism of precrossed modules $\beta : (Y_1, X_1, \delta_1) \longrightarrow (Y_2, X_2, \delta_2)$, making commutative the following diagram*

$$\begin{array}{ccccc} E_1 : (A, B, f) & \rightarrowtail & (Y_1, X_1, \delta_1) & \twoheadrightarrow & (M, P, \mu) \\ \| & & \downarrow \beta & & \| \\ E_2 : (A, B, f) & \rightarrowtail & (Y_2, X_2, \delta_2) & \twoheadrightarrow & (M, P, \mu) \end{array}$$

We will denote the congruence of extensions by $E_1 \equiv E_2$.

Observe that a morphism β making this diagram commutative, is always an isomorphism. We will denote by $Cext((M,P,\mu),(A,B,f))$ the quotient set of the central extensions of (M,P,μ) by (A,B,f), by the equivalence relation of congruence.

LEMMA 7. *Let* $(N,Q,\omega) \overset{i}{\rightarrowtail} (L,C,\omega) \overset{k}{\twoheadrightarrow} (M,P,\mu)$ *be an extension of precrossed modules, and let* (A,B,f) *be an abelian precrossed module. Then*

i) For each morphism of precrossed modules

$$\alpha = (\alpha_1, \alpha_2) : \left(\frac{N}{[C,N]\,[Q,L]\,[N,L]}, \frac{Q}{[Q,C]}, \overline{\omega} \right) \longrightarrow (A,B,f)$$

there exists a unique up to congruence central extension E_α *of* (M,P,μ) *by* (A,B,f)*, making the following diagram commutative*

$$\begin{array}{ccccccc} & (N,Q,\omega) & \overset{i}{\rightarrowtail} & (L,C,\omega) & \overset{k}{\twoheadrightarrow} & (M,P,\mu) \\ & \overline{\alpha}\downarrow & & \widetilde{\alpha}\downarrow & & \| \\ E_\alpha : & (A,B,f) & \overset{i_\alpha}{\rightarrowtail} & (Y_\alpha, X_\alpha, \delta) & \overset{k_\alpha}{\twoheadrightarrow} & (M,P,\mu) \end{array}$$

where $\overline{\alpha} = (\overline{\alpha_1}, \overline{\alpha_2})$ *and* $\widetilde{\alpha}$ *are induced by* α.

ii) For every two morphisms

$$\alpha, \alpha' : \left(\frac{N}{[C,N]\,[Q,L]\,[N,L]}, \frac{Q}{[Q,C]}, \overline{\omega} \right) \longrightarrow (A,B,f)$$

$E_\alpha \equiv E_{\alpha'}$ *if and only if, there exists a morphism of precrossed modules* $\varepsilon : (L,C,\omega) \to (A,B,f)$ *with*

$$\overline{\alpha'} = \overline{\alpha} + \varepsilon_{|(N,Q,\omega)} : (N,Q,\omega) \to (A,B,f)$$

Proof.

It is parallel to the proof of [5, Lemma 16].

THEOREM 8. *Let* (M,P,μ) *be a precrossed module, and* (A,B,f) *an abelian precrossed module. There exists a natural bijection*

$$Cext((M,P,\mu),(A,B,f)) \cong H^2((M,P,\mu),(A,B,f))$$

Proof.

It is analogous to the proof of [5, Theorem 17], using Lemma 7, and Corollary 2.

REMARK 9. *As follows from Theorem 8, one can define an abelian group structure on the quotient set* $Cext((M,P,\mu),(A,B,f))$*, such that the bijection* $Cext((M,P,\mu),(A,B,f)) \cong H^2((M,P,\mu),(A,B,f))$ *becomes an isomorphism of abelian groups. We will call Baer sum this operation between congruence classes of central extensions. The Baer sum also has the following independent description: given two central extensions of* (M,P,μ) *by* (A,B,f)*,*

$$E_i : (A,B,f) \rightarrowtail (Y_i,X_i,\delta_i) \overset{\varphi_i}{\twoheadrightarrow} (M,P,\mu)$$

where $i = 1,2$*, consider the pullback of* φ_1 *and* φ_2

$$\begin{array}{ccc} (Y_1 \underset{M}{\times} Y_2, X_1 \underset{P}{\times} X_2, \delta_1 \times \delta_2) & \longrightarrow & (Y_2,X_2,\delta_2) \\ \downarrow & & \downarrow \varphi_2 \\ (Y_1,X_1,\delta_1) & \overset{\varphi_1}{\longrightarrow} & (M,P,\mu) \end{array}$$

Take the coequalizer (Y,X,δ) *of the two canonical inclusions* $(A,B,f) \cong (A\times 0, B\times 0, f\times 0) \hookrightarrow (Y_1 \underset{M}{\times} Y_2, X_1 \underset{P}{\times} X_2, \delta_1 \times \delta_2)$ *and* $(A,B,f) \cong (0\times A, 0\times B, 0\times f) \hookrightarrow (Y_1 \underset{M}{\times} Y_2, X_1 \underset{P}{\times} X_2, \delta_1 \times \delta_2)$*. The Baer sum of* $\overline{E_1}$ *and* $\overline{E_2}$ *is the congruence class of the induced central extension*

$$(A,B,f) \rightarrowtail (Y,X,\delta) \twoheadrightarrow (M,P,\mu)$$

EXAMPLE 10. *Given a group* G *and an abelian group* A*, the group of congruence classes of central extensions of* G *by* A*,* $Cext(G,A)$*, is isomorphic to the group* $Cext((1,G,i),(1,A,i))$*. On the other hand, the Eilenberg-MacLane's cohomology group* $H^2(G,A)$ *is isomorphic to* $H^2((1,G,i),(1,A,i))$ *[1, Theorem 4.1]. Therefore, we deduce from Theorem 8, the classical theorem of classification of central extensions of groups (see, for example [9])*

$$Cext(G,A) \cong H^2(G,A)$$

It is proved in [5, Theorem 17] that for a crossed module (T,G,∂) and an abelian crossed module (A,B,f), the cohomology group $H^2_{CCG}((T,G,\partial),(A,B,f))$ of crossed modules is isomorphic to the group $Cext_{\mathcal{CM}}((T,G,\partial),(A,B,f))$ of congruence classes of crossed module central extensions of (T,G,∂) by (A,B,f)

$$(A,B,f) \rightarrowtail (S,H,\theta) \twoheadrightarrow (T,G,\partial)$$

So $H^2_{CCG}\left((T,G,\partial),(A,B,f)\right)$ is the subgroup of $H^2\left((T,G,\partial),(A,B,f)\right)$ corresponding to the classes of precrossed module central extensions of (T,G,∂) by (A,B,f) which are extensions of crossed modules.

Actually more is true: if we consider the adjunction $cr \dashv i$,

$$\begin{array}{ccc} \mathcal{PCM} & (M,P,\mu) & (T,G,\partial) \\ cr\downarrow\uparrow i & cr\downarrow & \uparrow i \\ \mathcal{CM} & (M,P,\mu)_{cr} & (T,G,\partial) \end{array}$$

where $i:\mathcal{CM}\to\mathcal{PCM}$ is the inclusion, and since i preserves surjectives then cr maps projectives to projectives. So, if $(V,R,\tau)\rightarrowtail(W,F,\tau)\overset{\pi}{\twoheadrightarrow}(T,G,\partial)$ is a projective presentation in $\mathcal{PCM}$ then $(V/\langle W,W\rangle,R,\overline{\tau})\rightarrowtail(W/\langle W,W\rangle,F,\overline{\tau})\overset{\xi}{\twoheadrightarrow}(T,G,\partial)$ is a projective presentation in $\mathcal{CM}$.

PROPOSITION 11. *Let (T,G,∂) be a crossed module, and (A,B,f) an abelian precrossed module. Let $(V,R,\tau)\rightarrowtail(W,F,\tau)\overset{\pi}{\twoheadrightarrow}(T,G,\partial)$ be a projective presentation in $\mathcal{PCM}$. Then $H^2_{CCG}\left((T,G,\partial),(A,B,f)\right)\cong$*

$$Ker(\xi^*:H^2((T,G,\partial),(A,B,f))\to H^2((W/\langle W,W\rangle,F,\overline{\tau}),(A,B,f)))$$

where $\xi:(W/\langle W,W\rangle,F,\overline{\tau})\to(T,G,\partial)$ is the morphism in $\mathcal{CM}$ induced by π.

Proof.

Let $\overline{E}\in H^2((T,G,\partial),(A,B,f))$. Then $\xi^*(\overline{E})=\overline{E^\xi}$ is defined by the pullback

$$\begin{array}{ccccc} E^\xi:(A,B,f) & \rightarrowtail & (Y,X,\delta)^\xi & \twoheadrightarrow & (W/\langle W,W\rangle,F,\overline{\tau}) \\ \| & & \downarrow & & \downarrow\xi \\ E:(A,B,f) & \rightarrowtail & (Y,X,\delta) & \twoheadrightarrow & (T,G,\partial) \end{array}$$

If $\xi^*(\overline{E})=0$, that is, E^ξ splits, then $(Y,X,\delta)^\xi\cong(A,B,f)\rtimes(W/\langle W,W\rangle,F,\overline{\tau})$ is a crossed module and its epimorphic image (Y,X,δ) is also a crossed module. Conversely, if $\overline{E}\in Cext_{\mathcal{CM}}((T,G,\partial),(A,B,f))$ then $(Y,X,\delta)^\xi\subset(Y,X,\delta)\times(W/\langle W,W\rangle,F,\overline{\tau})$ is a crossed module. Since $(W/\langle W,W\rangle,F,\overline{\tau})$ is a projective crossed module, $(Y,X,\delta)^\xi\twoheadrightarrow(W/\langle W,W\rangle,F,\overline{\tau})$ admits a splitting.

To each precrossed module (M,P,μ), we can associate a crossed module $(M,M\rtimes P,i)$, where i is the canonical inclusion of M into $M\rtimes P$. There is an isomorphism of groups between the group $Cext((M,P,\mu),(A,1,1))$ of congruence classes of precrossed modules central extensions of (M,P,μ) by $(A,1,1)$, and the group $Cext_{\mathcal{CM}}((M,M\rtimes P,i),(A,1,1))$

of congruence classes of crossed module central extensions of $(M, M\rtimes P, i)$ by $(A, 1, 1)$

$$Cext((M,P,\mu),(A,1,1)) \cong Cext_{\mathcal{CM}}((M, M\rtimes P, i),(A,1,1))$$

defined by the following correspondence: we assign to the class of a central extension of precrossed modules

$$\begin{array}{ccccc} A & \rightarrowtail & Y & \overset{\pi}{\twoheadrightarrow} & M \\ 1\downarrow & & \delta\downarrow & & \mu\downarrow \\ 1 & \rightarrowtail & P & \overset{Id}{\twoheadrightarrow} & P \end{array}$$

the class of the crossed module central extension

$$\begin{array}{ccccc} A & \rightarrowtail & Y & \overset{\pi}{\twoheadrightarrow} & M \\ 1\downarrow & & \pi\times 1\downarrow & & i\downarrow \\ 1 & \rightarrowtail & M\rtimes P & \overset{Id}{\twoheadrightarrow} & M\rtimes P \end{array}$$

of $(M, M\rtimes P, i)$ by $(A, 1, 1)$, where the action in the crossed module $(Y, M\rtimes P, \pi\times 1)$ is given by ${}^{(\pi(y_1),p)}y_2 = y_1 \cdot {}^{p}y_2 \cdot y_1^{-1}$ for $y_1, y_2 \in Y$ and $p \in P$.

The cohomology group $H^2_{CCG}((M, M\rtimes P, i),(A,1,1))$ is isomorphic to the group $Cext_{\mathcal{CM}}((M, M\rtimes P, i),(A,1,1))$ [5, Theorem 17], and the cohomology group $H^2((M,P,\mu),(A,1,1))$ is isomorphic to the group $Cext((M,P,\mu),(A,1,1))$ (Theorem 8); so

COROLLARY 12. *For a precrossed module (M,P,μ) and an abelian group A, there are group isomorphisms*
$H^2((M,P,\mu),(A,1,1)) \cong Cext((M,P,\mu),(A,1,1)) \cong$
$\cong Cext_{\mathcal{CM}}((M, M\rtimes P, i),(A,1,1)) \cong H^2_{CCG}((M, M\rtimes P, i),(A,1,1))$

The theory of central extensions of precrossed modules is also related to the theory of relative central extensions of groups [17]. A *relative central extension* of an epimorphism of groups $(P, M\rtimes P)$ by an abelian group A [17], is an exact sequence of groups

$$0 \to A \to N \overset{\lambda}{\to} M\rtimes P \overset{s}{\to} P \to 1$$

where $(N, M\rtimes P, \lambda)$ is a crossed module, and the induced action of P on A is trivial. A congruence of relative central extensions of $(P, M\rtimes P)$ by A [17] is a crossed module morphism $(f, Id) : (N, M\rtimes P, \lambda) \to (N', M\rtimes P, \lambda')$ making the following diagram commutative

$$\begin{array}{ccccccccc} 0 \to & A & \to & N & \overset{\lambda}{\to} & M\rtimes P & \overset{s}{\to} & P & \to 1 \\ & \| & & \downarrow f & & \| & & \| & \\ 0 \to & A & \to & N' & \overset{\lambda'}{\to} & M\rtimes P & \overset{s}{\to} & P & \to 1 \end{array}$$

The set of congruence classes of relative central extensions of $(P, M \rtimes P)$ by A is denoted by $\mathcal{E}xt(P, M \rtimes P; A)$.

There exists an isomorphism between the groups $\mathcal{E}xt(P, M \rtimes P; A)$ and $Cext((M, P, \mu), (A, 1, 1))$: we can assign to the class of a central extension of precrossed modules

$$\begin{array}{ccccc} A & \rightarrowtail & Y & \stackrel{\pi}{\twoheadrightarrow} & M \\ 1 \downarrow & & \delta \downarrow & & \mu \downarrow \\ 1 & \rightarrowtail & P & \stackrel{Id}{\twoheadrightarrow} & P \end{array}$$

the class of the relative central extension

$$0 \to A \to Y \xrightarrow{\pi \times 1} M \rtimes P \to P \to 1$$

of $(P, M \rtimes P)$ by A, where the action of the crossed module $(Y, M \rtimes P, \pi \times 1)$ is given again by ${}^{(\pi(y_1), p)}y_2 = y_1 \cdot {}^{p}y_2 \cdot y_1^{-1}$ for $y_1, y_2 \in Y$ and $p \in P$.

It is proved in [17, Théorème 1] that the relative cohomology group $H^3(P, M \rtimes P; A)$ of Loday, associated to the epimorphism $s : M \rtimes P \twoheadrightarrow P$, $s(m, p) = p$ and the trivial P-module A is isomorphic to the group $\mathcal{E}xt(P, M \rtimes P; A)$.

There is also an isomorphism of groups between the cohomology group $H^2((M, P, \mu), (A, 1, 1))$ and the group $Cext((M, P, \mu), (A, 1, 1))$ (Theorem 8), so

COROLLARY 13. *For a precrossed module (M, P, μ) and an abelian group A, there are group isomorphisms*
$H^2((M, P, \mu), (A, 1, 1)) \cong Cext((M, P, \mu), (A, 1, 1)) \cong \mathcal{E}\mathrm{xt}(P, M \rtimes P; A)$
$\cong H^3(P, M \rtimes P; A)$

5. Universal Coefficient Theorem. Galois theory.

If G is a group, and C is an abelian group, regarded as trivial G-module, then the Universal Coefficient Theorem [9, Theorem 15.1] states that for every $n \geq 1$ the sequence

$$0 \to Ext^1_{\mathbb{Z}}(H_{n-1}(G), C) \to H^n(G, C) \to Hom_{\mathbb{Z}}(H_n(G), C) \to 0$$

is exact and natural. Next we establish the corresponding result for precrossed module cohomology.

THEOREM 14. *Let (M,P,μ) be a precrossed module and let (A,B,f) be an abelian precrossed module.*

(i) There is a natural exact sequence

$0 \to Ext^1_{\mathcal{APCM}}(H_1(M,P,\mu),(A,B,f)) \to H^2((M,P,\mu),(A,B,f)) \to Hom_{\mathcal{APCM}}(H_2(M,P,\mu),(A,B,f)) \to Ext^2_{\mathcal{APCM}}(H_1(M,P,\mu),(A,B,f)) \to H^3((M,P,\mu),(A,B,f))$.

(ii) If $H_2(M,P,\mu)=0$, then there is a natural exact sequence

$0 \to Ext^2_{\mathcal{APCM}}(H_1(M,P,\mu),(A,B,f)) \to H^3((M,P,\mu),(A,B,f)) \to Hom_{\mathcal{APCM}}(H_3(M,P,\mu),(A,B,f))$

(iii) If $H_i(M,P,\mu)=0$ for all $1<i\leq n$ and $n\geq 3$, then
$H^i((M,P,\mu),(A,B,f))=0$ for all $3<i\leq n$ and
$H^{n+1}((M,P,\mu),(A,B,f)) \cong Hom_{\mathcal{APCM}}(H_{n+1}(M,P,\mu),(A,B,f))$.

(iv) Let $(A,B,f) \rightarrowtail \mathcal{I}^\bullet$ be an injective resolution of (A,B,f) in the category of abelian precrossed modules, such that $\mathcal{I}^m=0$ for all $m\geq 3$ (such a resolution exists [5]). If $Hom_{\mathcal{APCM}}(H_i(M,P,\mu),\mathcal{I}^2)=0$ for all $i\geq 1$, then there exists an exact and natural sequence

$0 \to Ext^1_{\mathcal{APCM}}(H_n(M,P,\mu),(A,B,f)) \to H^{n+1}((M,P,\mu),(A,B,f)) \to Hom_{\mathcal{APCM}}(H_{n+1}(M,P,\mu),(A,B,f)) \to 0$ *for all $n\geq 1$.*

Proof.

It is parallel to the proof of [5, Theorem 18].

Recall from [5] and [19, Corollary 10.10] that the category of abelian precrossed modules $\mathcal{APCM}$ has global dimension 2, and that the injective abelian precrossed modules are those of the form $(I\oplus J, I, p_1)$, where I and J are divisible abelian groups, and p_1 is the first projection from the coproduct.

EXAMPLE 15. *Given an injective resolution $A \overset{\alpha}{\rightarrowtail} I^0 \overset{\beta}{\to} I^1 \to 0\ldots$ of an abelian group A, and a group G, the abelian precrossed module $(1,A,i)$ has an injective resolution*

$$\begin{aligned}\mathcal{I}^\bullet \;:\; & (1,A,i) \overset{(1,\alpha)}{\rightarrowtail} (I^0,I^0,Id) \xrightarrow{(\{Id,\beta\},\beta)} (I^0\oplus I^1, I^1, p_2) \to \\ & \to (Coker\,\{Id,\beta\},1,1) \to 0\ldots\end{aligned}$$

such that for every $i\geq 1$, $Hom_{\mathcal{APCM}}(H_i(1,G,i),\mathcal{I}^2)=0$.

With this resolution it is easy to see that $Ext^1_{\mathcal{APCM}}(H_n(1,G,i),(1,A,i))$ is isomorphic to $Ext^1_{\mathbb{Z}}(H_n(G),A)$, for $n\geq 1$.

On the other hand, the group $H^{n+1}((1,G,i),(1,A,i))$ is isomorphic to $H^{n+1}(G,A)$, and the group $Hom_{\mathcal{APCM}}(H_{n+1}(1,G,i),(1,A,i))$ is isomorphic to $Hom_{\mathbb{Z}}(H_{n+1}(G),A)$, for $n\geq 0$.

From Theorem 14 (iv) we can deduce the exact and natural sequence of the Universal Coefficient Theorem for the cohomology of groups [9, *VI. Theorem 15.1]*

$$0 \to Ext^1_{\mathbb{Z}}(H_n(G), A) \to H^{n+1}(G, A) \to Hom_{\mathbb{Z}}(H_{n+1}(G), A) \to 0$$

for $n \geqslant 1$.

Remark that if a precrossed module (M, P, μ) is perfect [2], that is, it coincides with its commutator precrossed submodule, then $H_1(M, P, \mu) = 0$, and the exact sequence in Theorem 14 (i) provides an isomorphism

$$H^2((M, P, \mu), (A, B, f)) \cong Hom_{\mathcal{APCM}}(H_2(M, P, \mu), (A, B, f))$$

On the other hand, in Theorem 8, it is proved that $H^2((M, P, \mu), (A, B, f))$ classifies the central extensions of (M, P, μ) by an abelian precrossed module (A, B, f)

$$Cext((M, P, \mu), (A, B, f)) \cong H^2((M, P, \mu), (A, B, f))$$

So the category of the congruence classes of central extensions $Centr(M, P, \mu)$ of the perfect precrossed module (M, P, μ), is equivalent to the comma category $H_2(M, P, \mu) \downarrow \mathcal{APCM}$.

This equivalence is made as follows: if $(A, B, f) \rightarrowtail (Y, X, \delta) \overset{(\varphi,\varphi')}{\twoheadrightarrow} (M, P, \mu)$ is an extension of (M, P, μ), then there exists a unique morphism Δ, making commutative the following diagram

$$\begin{array}{ccccc} H_2(M, P, \mu) & \rightarrowtail & (M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu) & \overset{(\lambda,\zeta)}{\twoheadrightarrow} & (M, P, \mu) \\ \downarrow \Delta & & \downarrow & \nearrow_{(\varphi,\varphi')} & \\ (A, B, f) & \rightarrowtail & (Y, X, \delta) & & \end{array}$$

In [13], Janelidze and Kelly show how to describe the category of the central extensions of an object B, $Centr(B)$, with respect to a pair formed by an exact category $\mathcal{C}$ and an "admissible" subcategory $\mathcal{X}$ of $\mathcal{C}$, using a generalized Galois theory developed by Janelidze in [10], [11] and [12]. Following [13], from the universal central extension $(\lambda, \zeta) : (M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu) \twoheadrightarrow (M, P, \mu)$ we obtain an internal groupoid in $\mathcal{APCM}$, the *Galois pregroupoid Gal* $((M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu), (\lambda, \zeta))$, and an equivalence of categories

$$Centr(M, P, \mu) \simeq \{Gal\,((M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu), (\lambda, \zeta))\,, \mathcal{APCM}\}$$

where the category in the right is a certain full subcategory of the category of internal actions of $Gal\,((M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu), (\lambda, \zeta))$.

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