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The precise center of a crossed module

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Abstract. We generalize the definition of the precise center of a group to the crossed modules context. We construct the Ganea map for the homology of crossed modules, and we study the connections between the precise center of a crossed module and the Ganea map. We extend some other known notions from group theory such as capable and relatively capable groups, capable pairs and unicentral groups with the definitions of capable and unicentral crossed modules. Finally we show how to apply these constructions to solve some open questions in the theory of crossed modules.

1 Introduction

Originally the notions of *capable* group and *unicentral* group appeared separately. Unicentral groups were introduced in the late 1960s by Evens in [10]. A group G is called *unicentral* if the center of every central extension of G maps onto $\mathrm{Z}(G)$.

On the other hand, capable groups were first studied by Baer [2] in the late 1930s. A group is said to be capable if it is isomorphic to the group of inner automorphisms of some group. While a characterization of abelian finitely generated capable groups was already determined by Baer, investigation of capability for other classes of groups has received renewed attention in the last decade.

In [4], Beyl, Felgner and Schmid showed that there is a common approach to capable and unicentral groups. They define for a given group G a central subgroup $\mathrm{Z}^*(G)$ which is the smallest subject to being the image in G of the center of a central extension of G. Moreover, $\mathrm{Z}^*(G)$ is also the smallest central subgroup of G whose factor group is capable: also a group G is capable if and only if $\mathrm{Z}^*(G) = 1$ and G is unicentral if and only if $\mathrm{Z}^*(G) = \mathrm{Z}(G)$.

They also study a very interesting connection between the *precise center* of a group $\mathrm{Z}^*(G)$, the Schur multiplicator and the Ganea map: a central subgroup N of G satisfies $N \subset \mathrm{Z}^*(G)$ precisely when the mapping $H_2(G) \to H_2(G/N)$ is monomorphic. This property implies that the precise center $\mathrm{Z}^*(G)$ is the left kernel of the Ganea map $\gamma_G : \mathrm{Z}(G) \otimes G_{\mathrm{ab}} \to H_2(G)$.

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In this work we will extend this discussion on capability and unicentrality to the crossed modules context.

The algebraic study of the category of crossed modules was initiated by Norrie [15] in her thesis. She extended group-theoretic concepts and structures to crossed modules. For example, she defined for each crossed module (T, G, ∂) its *actor* $\mathrm{A}(T, G, \partial)$, a crossed module which generalizes the automorphism group of a group. She constructed a canonical morphism $(\eta, \gamma) : (T, G, \partial) \to \mathrm{A}(T, G, \partial)$ of crossed modules, and the *center* $\mathrm{Z}(T, G, \partial)$ of (T, G, ∂) was defined as the kernel of (η, γ), while its image is the *inner actor* $\mathrm{I}(T, G, \partial)$ of (T, G, ∂).

Recently Carrasco, Cegarra and R.-Grandjeán extended in [6] the Eilenberg–MacLane (co)homology of groups with their cotriple (co)homology of crossed modules. For each crossed module (T, G, ∂), abelian crossed module (A, B, f) and for $n \geqslant 1$ they define the *n-th homology crossed module* $H_n(T, G, \partial)$ *of* (T, G, ∂), and the *n-th cohomology group* $H^n((T, G, \partial), (A, B, f))$ *of* (T, G, ∂) *with coefficients in* (A, B, f).

Furthermore, they generalized some classical results from the homology of groups. We will be specially interested in two of them, which are generalizations of key results for the treatment of several topics in group theory: the construction of a five-term exact sequence for the homology of crossed modules, and a Hopf formula for the second homology of a crossed module.

In [16], Pirashvili introduced the tensor product of two abelian crossed modules, and he used it to construct the *Ganea term*, that is, a sixth term which extends the five-term exact sequence

$$(P, N, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} \to H_2(T, G, \partial) \to H_2(U, Q, \omega) \to (P, N, \partial)$$
$$\to H_1(T, G, \partial) \to H_1(U, Q, \omega) \to 0$$

associated to a central extension of crossed modules

$$(P, N, \partial) \rightarrowtail (T, G, \partial) \twoheadrightarrow (U, Q, \omega).$$

We will use all of these crossed module constructions to develop our notion of the precise center of a crossed module.

We begin in Section 2 by recalling known facts about the category of crossed modules and the definition of the homology, which can be found in detail in [6] or in [15]. We deduce some basic properties of projective crossed modules which will be used later.

In Section 3 we describe explicitly how the Ganea map

$$(P, N, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} \to H_2(T, G, \partial)$$

introduced by Pirashvili in [16] is defined.

In Section 4 we introduce unicentral and capable crossed modules. In analogy with capable and unicentral groups, we say that a crossed module (T, G, ∂) is capable if it is isomorphic to the inner actor of some crossed module, and that (T, G, ∂) is unicentral if the center of every central extension of (T, G, ∂) maps onto $Z(T, G, \partial)$. We also define the precise center $Z^*(T, G, \partial)$ of a crossed module, which is seen to generalize the precise center of a group. It has similar properties to those of the precise center of a group, as was remarked above; then a crossed module (T, G, ∂) is capable if and only if $Z^*(T, G, \partial) = 0$, and is unicentral if and only if $Z^*(T, G, \partial) = Z(T, G, \partial)$.

We apply these properties to find out in which cases the typical examples of crossed modules are capable or unicentral crossed modules.

In Section 5 we analyze the connections between the precise center of a crossed module and the Ganea map. The main result of this section shows that a central crossed submodule (P, N, ∂) of (T, G, ∂) satisfies $(P, N, \partial) \subset Z^*(T, G, \partial)$ if and only if the Ganea map $(P, N, \partial) \otimes (T, G, \partial)_{ab} \to H_2(T, G, \partial)$ is zero. This property will provide us with some computable expressions for $Z^*(T, G, \partial)$.

In Section 6 we study connections between capable crossed modules and some other capability notions developed in [9] and [17], like *capable pairs of groups* or *relatively capable groups*.

Finally, in Section 7, we apply all of these constructions to solve some open questions and prove results concerning perfect crossed modules which were conjectured by Norrie in [15].

2 Projective crossed modules and the homology of crossed modules

A *precrossed module* (T, G, ∂) is a group homomorphism $\partial : T \to G$ together with an action of G on T, denoted by ${}^g t$ for $t \in T$ and $g \in G$, and satisfying $\partial({}^g t) = g\partial(t)g^{-1}$. We call (T, G, ∂) a *crossed module* if in addition ${}^{\partial(t)}t' = tt't^{-1}$ for $t, t' \in T$.

A *crossed module morphism* $(\Phi, \Psi) : (T_1, G_1, \partial_1) \to (T_2, G_2, \partial_2)$ is a pair of group homomorphisms $\Phi : T_1 \to T_2$ and $\Psi : G_1 \to G_2$ such that $\Psi \circ \partial_1 = \partial_2 \circ \Phi$ and $\Phi({}^g t) = {}^{\Psi(g)}\Phi(t)$ for all $t \in T_1$ and $g \in G_1$. We say that (Φ, Ψ) is *injective* (*surjective*) when both Φ and Ψ are injective (surjective) homomorphisms.

We denote the category of crossed modules by $\mathcal{CM}$.

A crossed module (T, G, ∂) is called *aspherical* if $\mathrm{Ker}(\partial) = 0$. We call it *simply connected* if $\mathrm{Im}(\partial) = G$.

Example 1. (1) Let N be a normal subgroup of G. The inclusion homomorphism $i : N \hookrightarrow G$ with the action ${}^g n = gng^{-1}$, $g \in G$, $n \in N$, is a crossed module. Every aspherical crossed module is isomorphic to a crossed module of the form (N, G, i).

(2) In particular, (G, G, Id) and $(1, G, i)$ are crossed modules.

(3) $(A, G, 0)$ is a crossed module where A is an ordinary $\mathbb{Z}G$-module.

(4) Every central extension of groups $K \rightarrowtail T \overset{\partial}{\twoheadrightarrow} G$ gives rise to a simply connected crossed module (T, G, ∂).

A *crossed submodule* (N, Q, ∂') of a crossed module (T, G, ∂) is a crossed module such that N and Q are respectively subgroups of T and G, the action of Q on N is induced by the action of G on T and $\partial|_N = \partial'$. We call (N, Q, ∂') a *normal crossed submodule* if in addition Q is normal in G, ${}^g n \in N$ and ${}^q t t^{-1} \in N$ for all $n \in N$, $q \in Q$, $t \in T$ and $g \in G$.

If (N, Q, ∂) is a normal crossed submodule of (T, G, ∂), we define the *quotient crossed module* $(T, G, \partial)/(N, Q, \partial)$ as $(T/N, G/Q, \bar{\partial})$ where the homomorphism $\bar{\partial}$ is induced by ∂ and G/Q acts on T/N by ${}^{gQ}tN = ({}^{g}t)N$ for $t \in T$ and $g \in G$.

The *center* $Z(T, G, \partial)$ of a crossed module (T, G, ∂) is the normal crossed submodule $(T^G, \mathrm{St}_G(T) \cap Z(G), \partial)$, where $T^G = \{t \in T \mid {}^{g}t = t \text{ for all } g \in G\}$, $\mathrm{St}_G(T)$ denotes the group $\{g \in G \mid {}^{g}t = t \text{ for all } t \in T\}$ and $Z(G)$ is the center of the group G.

A crossed module (T, G, ∂) is said to be *abelian* if $(T, G, \partial) = Z(T, G, \partial)$. Equivalently T and G are abelian groups and G acts trivially on T. We denote the category of abelian crossed modules by $\mathscr{ACM}$.

If (N, Q, ∂) is a normal crossed submodule of (T, G, ∂), we define the *commutator crossed submodule* $[(T, G, \partial), (N, Q, \partial)]$ of (T, G, ∂) and (N, Q, ∂), as the normal crossed submodule $([G, N][Q, T], [G, Q], \partial)$ of (T, G, ∂), where $[G, N][Q, T]$ denotes the normal subgroup of T generated by the set of elements

$$\{{}^{g}nn^{-1}, {}^{q}tt^{-1} \mid n \in N, t \in T, g \in G \text{ and } q \in Q\}$$

and $[G, Q]$ is the usual commutator subgroup of G with Q.

We call the abelian crossed module

$$(T, G, \partial)_{\mathrm{ab}} = \frac{(T, G, \partial)}{[(T, G, \partial), (T, G, \partial)]}$$

the *abelianization* of (T, G, ∂).

In [6], Carrasco, Cegarra and R.-Grandjeán proved that the category of crossed modules is an *algebraic* category, that is, that there is a tripleable forgetful functor from the category $\mathscr{CM}$ to the category of sets $\mathscr{S}et$.

Theorem 2 ([6]). *The forgetful functor $\mathscr{U} : \mathscr{CM} \to \mathscr{S}et$, $\mathscr{U}(T, G, \partial) = T \times G$ which assigns to each crossed module (T, G, ∂) the cartesian product of the underlying sets T and G is tripleable. Moreover $\mathscr{U}$ has as left adjoint the functor $\mathscr{F} : \mathscr{S}et \to \mathscr{CM}$, $\mathscr{F}(X) = (\overline{FX}, FX * FX, i)$, where FX denotes the free group over the set X, $\overline{FX}$ is the kernel of the projection $\langle 0, \mathrm{Id} \rangle$ of the free product $FX * FX$ of groups onto the second factor, and i denotes the inclusion homomorphism.*

In consequence every free crossed module, i.e. every object $\mathscr{F}(X)$, is a projective object in the category $\mathscr{CM}$, and so $\mathscr{CM}$ has enough projective objects since every crossed module admits a projective presentation as a quotient of a free crossed module by means of the counit of the adjunction between $\mathscr{F}$ and $\mathscr{U}$.

We deduce the following properties of the projective crossed modules:

Proposition 3. *Let (Y, F, μ) be a projective crossed module. Then*

(a) *(Y, F, μ) is aspherical,*

(b) *Y, F and F/Y are free groups, and*

(c) $Y \cap [F, F] = [F, Y]$.

Proof. Take the presentation of (Y, F, μ) as a quotient of a free crossed module

$$(\overline{FX}, FX * FX, i) \xrightarrow{(\pi, \pi')} (Y, F, \mu).$$

Since (Y, F, μ) is projective, the morphism (π, π') splits. Then (Y, F, μ) is isomorphic to a crossed submodule of $(\overline{FX}, FX * FX, i)$, μ is injective and F and Y are free groups.

If we take the cokernels of μ and i then in the diagram

$$\begin{array}{ccccc}
Y & \xrightarrow{\ \mu\ } & F & \xrightarrow{\ \mu^c\ } & F/Y \\
\downarrow & & \downarrow & & \downarrow \\
\overline{FX} & \xrightarrow{\ i\ } & FX * FX & \xrightarrow{\langle 0, \mathrm{Id}\rangle} & FX \\
{\scriptstyle \pi}\downarrow & & {\scriptstyle \pi'}\downarrow & & \downarrow \\
Y & \xrightarrow{\ \mu\ } & F & \xrightarrow{\ \mu^c\ } & F/Y
\end{array}$$

the composites of the morphisms in the two first columns are identities, and so the composite in the third column also gives the identity. It follows that F/Y is a subgroup of the free group FX, and so it is a free group. The Schur multiplicator of F/Y is zero and (c) is true. □

The pair of adjoint functors $\mathscr{F}$ and $\mathscr{U}$ induces a cotriple (C, δ, ε) in $\mathscr{CM}$, and following the general theory of cotriple homology of Barr and Beck [3], Carrasco, Cegarra and R.-Grandjeán defined the *homology crossed modules* $H_n(T, G, \partial)$ of a crossed module (T, G, ∂) as the derived functors of the abelianization functor $\mathrm{ab} : \mathscr{CM} \to \mathscr{ACM}$.

3 Ganea map for the homology of crossed modules

In [16], Pirashvili defined a tensor product of abelian crossed modules:

Definition 1. The *tensor product* of two abelian crossed modules (A, B, f) and (C, D, g) is the abelian crossed module

$$(A, B, f) \otimes (C, D, g) = (\mathrm{Coker}(\alpha), B \otimes D, \bar{\delta}),$$

where $\alpha = \{f \otimes \mathrm{Id}, -\mathrm{Id} \otimes g\}$ and $\bar{\delta}$ is induced on $\mathrm{Coker}(\alpha)$ by the homomorphism $\delta = \langle \mathrm{Id} \otimes g, f \otimes \mathrm{Id} \rangle$:

$$\begin{array}{ccccc}
A \otimes C & \xrightarrow{\ \alpha\ } & (B \otimes C) \oplus (A \otimes D) & \xrightarrow{\ \alpha^c\ } & \mathrm{Coker}(\alpha). \\
 & & {\scriptstyle \delta}\downarrow & \swarrow {\scriptstyle \bar{\delta}} & \\
 & & B \otimes D & &
\end{array}$$

For a central extension of crossed modules

$$(P, N, \partial) \rightarrowtail (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$$

Carrasco, Cegarra and R.-Grandjeán in [6] and Ladra and R.-Grandjeán in [13] constructed five-term exact sequences for the homology of crossed modules

$$\begin{aligned} H_2(T, G, \partial) \xrightarrow{(\sigma,\sigma')} H_2(U, Q, \omega) \longrightarrow (P, N, \partial) \longrightarrow H_1(T, G, \partial) \\ \longrightarrow H_1(U, Q, \omega) \longrightarrow 0. \end{aligned}$$

Pirashvili announces in [16] the existence of a low-dimensional homology exact sequence which extends the five-term exact sequence of Carrasco, Cegarra and R.-Grandjeán:

Theorem 4 (Ganea map for the homology of crossed modules). *Let*

$$(P, N, \partial) \rightarrowtail (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$$

be a central extension of crossed modules. There exists a natural morphism of abelian crossed modules

$$\chi_{(P,N,\partial)} : (P, N, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} \to H_2(T, G, \partial)$$

which extends the five-term exact sequence for the homology of crossed modules:

$$\begin{aligned} (P, N, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} \to H_2(T, G, \partial) \to H_2(U, Q, \omega) \to (P, N, \partial) \\ \to H_1(T, G, \partial) \to H_1(U, Q, \omega) \to 0. \end{aligned}$$

Next we give an explicit construction of this Ganea map.

Remark 5. Given a projective presentation of (T, G, ∂)

$$(V, R, \mu) \rightarrowtail (Y, F, \mu) \xrightarrow{(\pi,\pi')} (T, G, \partial)$$

there is a commutative diagram of extensions

$$\begin{array}{ccccc} & \nearrow & (W, S, \mu) & \twoheadrightarrow & (P, N, \partial) \\ & & \downarrow & & \downarrow \\ (V, R, \mu) & \rightarrowtail & (Y, F, \mu) & \xrightarrow{(\pi,\pi')} & (T, G, \partial) \\ & & & \searrow & \downarrow \\ & & & & (U, Q, \omega) \end{array}$$

making (Y, F, μ) a projective presentation of (U, Q, ω), where (W, S, μ) is the inverse image of (P, N, ∂) under (π, π'). Using Hopf's formula for the second homology of a crossed module (see [6]) we can express the morphism

$$(\sigma, \sigma') : H_2(T, G, \partial) \to H_2(U, Q, \omega)$$

as

$$\begin{array}{ccc} \dfrac{V \cap [F, Y]}{[F, V][R, Y]} & \xrightarrow{\sigma} & \dfrac{W \cap [F, Y]}{[F, W][S, Y]} \\ {\scriptstyle \bar{\mu}}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{\mu}} \\ \dfrac{R \cap [F, F]}{[F, R]} & \xrightarrow{\sigma'} & \dfrac{S \cap [F, F]}{[F, S]}. \end{array}$$

From now on we will consider the homomorphism μ of the aspherical crossed module (Y, F, μ) as an inclusion, and we will identify the elements of Y with their images in F.

Lemma 6. $[(Y, F, \mu), (W, S, \mu)] \subset (V, R, \mu)$; *that is,* $[F, S] \subset R$, $[F, W] \subset V$ *and* $[S, Y] \subset V$.

This follows since $[(P, N, \partial), (T, G, \partial)] = 0$.

Proof of Theorem 4. Since

$$\mathrm{Ker}(\sigma, \sigma') = \left(\frac{[F, W][S, Y]}{[F, V][R, Y]}, \frac{[F, S]}{[F, R]}, \bar{\mu} \right)$$

we try to define a natural surjective morphism (γ, γ')

$$\begin{array}{ccc} \mathrm{Coker}(\alpha) & \xrightarrow{\gamma} & \dfrac{[F, W][S, Y]}{[F, V][R, Y]} \\ {\scriptstyle \bar{\delta}}\Big\downarrow & & \Big\downarrow{\scriptstyle \bar{\mu}} \\ N \otimes G_{\mathrm{ab}} & \xrightarrow{\gamma'} & \dfrac{[F, S]}{[F, R]}. \end{array}$$

The second component γ' is up to inclusion the classical Ganea map for the integral homology of groups associated to the central subgroup N of G (a construction can be found in [8])

$$N \otimes G_{\mathrm{ab}} \cong \frac{S}{R} \otimes \frac{F}{[F, F]R} \to H_2(G) \cong \frac{R \cap [F, F]}{[F, R]}, \quad \bar{s} \otimes \bar{f} \mapsto \overline{[f, s]}.$$

Now we construct γ, which will be induced on $\operatorname{Coker}(\alpha)$ by a homomorphism $\langle\gamma_1, \gamma_2\rangle$:

$$\begin{array}{ccccc} P \otimes \dfrac{T}{[G,T]} & \xrightarrow{\alpha} & \left(N \otimes \dfrac{T}{[G,T]}\right) \oplus (P \otimes G_{\mathrm{ab}}) & \xrightarrow{\alpha^c} & \operatorname{Coker}(\alpha). \\ & & \Big\downarrow \langle\gamma_1,\gamma_2\rangle & \swarrow \gamma & \\ & & \dfrac{[F,W][S,Y]}{[F,V][R,Y]} & & \end{array}$$

First, we obtain the homomorphism

$$\gamma_1 : N \otimes \frac{T}{[G,T]} \cong \frac{S}{R} \otimes \frac{Y}{[F,Y]V} \to \frac{[F,W][S,Y]}{[F,V][R,Y]}, \quad \bar{s} \otimes \bar{y} \mapsto \overline{[y,s]}$$

from the map

$$c_1 : S \times Y \to \frac{[F,W][S,Y]}{[F,V][R,Y]}, \quad c_1(s,y) = \overline{[y,s]},$$

because c_1 satisfies

$$c_1(ss', y) = c_1(s,y)c_1(s',y), \quad c_1(s, yy') = c_1(s,y)c_1(s,y'),$$
$$c_1(S \times [F,Y]) = 0, \quad c_1(S \times V) = 0, \quad c_1(R \times Y) = 0$$

for all $s, s' \in S$ and $y, y' \in Y$. We will check the equalities $c_1(ss', y) = c_1(s,y)c_1(s',y)$ and $c_1(s, yy') = c_1(s,y)c_1(s,y')$; the others are clear:

$$c_1(ss', y) = \overline{[y, ss']} = \overline{[y,s]s[y,s']s^{-1}} = \overline{[y,s]} \cdot \overline{s[y,s']s^{-1}} = \overline{[y,s][y,s']},$$

since $[S,[S,Y]] \subset [S,V] \subset [F,V]$ by Lemma 6, and

$$c_1(s, yy') = \overline{[yy', s]} = \overline{y[y',s]y^{-1}[y,s]} = \overline{y[y',s]y^{-1}} \cdot \overline{[y,s]} = \overline{[y,s][y',s]},$$

since $([F,W][S,Y])/([F,V][R,Y])$ is an abelian group, and

$$[Y,[S,Y]] \subset [Y,V] \subset [F,V]$$

by Lemma 6.

The homomorphism

$$\gamma_2 : P \otimes G_{\mathrm{ab}} \cong \frac{W}{V} \otimes \frac{F}{[F,F]R} \to \frac{[F,W][S,Y]}{[F,V][R,Y]}, \quad \overline{w} \otimes \bar{f} \mapsto \overline{[f,w]}$$

is obtained from the map

$$c_2 : W \times F \to \frac{[F, W][S, Y]}{[F, V][R, Y]}, \quad c_2(w, f) = \overline{[f, w]},$$

because c_2 satisfies

$$c_2(ww', f) = c_2(w, f)c_2(w', f), \quad c_2(w, ff') = c_2(w, f)c_2(w, f')$$
$$c_2(W \times [F, F]) = 0, \quad c_2(W \times R) = 0, \quad c_2(V \times F) = 0$$

for all $w, w' \in W$ and $f, f' \in F$.

The induced homomorphism $\gamma = \langle \gamma_1, \gamma_2 \rangle$ in the coproduct

$$\left(N \otimes \frac{T}{[G, T]}\right) \oplus (P \otimes G_{\mathrm{ab}}) \overset{\gamma}{\twoheadrightarrow} \frac{[F, W][S, Y]}{[F, V][R, Y]}$$

is clearly surjective, and the pair (γ, γ') is easily seen to be a crossed module morphism. The Ganea map $\chi_{(P,N,\partial)}$ is defined as the composite of (γ, γ') with the inclusion map

$$\left(\frac{[F, W][S, Y]}{[F, V][R, Y]}, \frac{[F, S]}{[F, R]}, \bar{\mu}\right) \to H_2(T, G, \partial).$$

To prove that $\chi_{(P,N,\partial)}$ is a natural morphism which does not depend on the chosen projective presentation, we take a morphism of central extensions

$$\begin{array}{ccccc} E : (P, N, \partial) & \rightarrowtail & (T, G, \partial) & \twoheadrightarrow & (U, Q, \omega) \\ \downarrow & & \downarrow{\scriptstyle (t,t')} & & \downarrow \\ \tilde{E} : (\tilde{P}, \tilde{N}, \tilde{\partial}) & \rightarrowtail & (\tilde{T}, \tilde{G}, \tilde{\partial}) & \twoheadrightarrow & (\tilde{U}, \tilde{Q}, \tilde{\omega}) \end{array}$$

and projective presentations of E and $\tilde{E}$ as in Remark 5. An easy calculation yields the commutativity of the following diagram:

$$\begin{array}{ccc} (P, N, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} & \xrightarrow{\chi_{(P,N,\partial)}} & H_2(T, G, \partial) \\ {\scriptstyle (t,t') \otimes (t,t')_{\mathrm{ab}}}\downarrow & & \downarrow{\scriptstyle H_2(t,t')} \\ (\tilde{P}, \tilde{N}, \tilde{\partial}) \otimes (\tilde{T}, \tilde{G}, \tilde{\partial})_{\mathrm{ab}} & \xrightarrow{\chi_{(\tilde{P},\tilde{N},\tilde{\partial})}} & H_2(\tilde{T}, \tilde{G}, \tilde{\partial}). \end{array}$$

Taking $E = \tilde{E}$ and $(t, t') = \mathrm{Id}$, we conclude that $\chi_{(P,N,\partial)}$ is independent of the chosen projective presentation, since $- \otimes -$ and $H_2(-)$ are functors. □

Remark 7. The Ganea map for crossed modules provides the classical Ganea map for groups [8] when groups are considered as crossed modules in the usual ways.

In general, the Ganea map for crossed modules is different from the Ganea map for the homology of precrossed modules constructed in [1], even in the simplest examples. For example the Ganea map associated to the group extension $\mathbb{Z} \rightarrowtail \mathbb{Z} \twoheadrightarrow 0$ is the zero map, since $H_2(\mathbb{Z}) = 0$, the Ganea map associated to the crossed module extension $(\mathbb{Z}, \mathbb{Z}, \mathrm{Id}) \rightarrowtail (\mathbb{Z}, \mathbb{Z}, \mathrm{Id}) \twoheadrightarrow (0, 0, \mathrm{Id})$ is also the zero map, but considered as a precrossed module extension its Ganea map for the homology of precrossed modules is a surjective morphism $(\mathbb{Z}^3, \mathbb{Z}, \delta) \twoheadrightarrow (\mathbb{Z}, 0, 0)$; see [1].

4 Capable and unicentral crossed modules

We shall introduce some kinds of crossed modules, like the unicentral and capable crossed modules, which will be related to the Ganea map.

Definition 2. A crossed module (U, Q, ω) is called *unicentral* if every central extension $(\varphi, \varphi') : (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$ satisfies $(\varphi, \varphi')(\mathrm{Z}(T, G, \partial)) = \mathrm{Z}(U, Q, \omega)$:

$$\begin{array}{ccccc} & \nearrow & \mathrm{Z}(T, G, \partial) & \twoheadrightarrow & \mathrm{Z}(U, Q, \omega) \\ & & \downarrow & & \downarrow \\ \mathrm{Ker}(\varphi, \varphi') & \rightarrowtail & (T, G, \partial) & \xrightarrow{(\varphi, \varphi')} & (U, Q, \omega). \end{array}$$

Definition 3. A crossed module (U, Q, ω) is called *capable* if there is a crossed module central extension

$$\mathrm{Z}(T, G, \partial) \rightarrowtail (T, G, \partial) \xrightarrow{(\varphi, \varphi')} (U, Q, \omega).$$

Remark 8. The *inner actor* of a crossed module (T, G, ∂) was defined by Norrie [15] as the quotient

$$\mathrm{I}(T, G, \partial) = \frac{(T, G, \partial)}{\mathrm{Z}(T, G, \partial)}.$$

Therefore a crossed module is capable if and only if it is the inner actor of another crossed module.

Example 9. A group G is called *capable* if it is the group of inner automorphisms $Q/\mathrm{Z}(Q)$ of some group Q; see [5]. Therefore G is capable if and only if $(1, G, i)$ or (G, G, Id) are capable crossed modules.

Definition 4. We define the *precise center* $\mathrm{Z}^*(U, Q, \omega)$ of a crossed module (U, Q, ω) to be the intersection of the crossed submodules $(\varphi, \varphi')(\mathrm{Z}(T, G, \partial))$, where $(\varphi, \varphi') : (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$ is a central extension of (U, Q, ω).

Remark 10. (1) Note that in the definition of $\mathrm{Z}^*(U, Q, \omega)$, each (φ, φ') is surjective, so $\mathrm{Z}^*(U, Q, \omega)$ is an intersection of central crossed submodules and then it is a central and normal crossed submodule of (U, Q, ω).

(2) From this definition we deduce that a crossed module (U, Q, ω) is unicentral if and only if $Z^*(U, Q, \omega) = Z(U, Q, \omega)$. Next, we see that a crossed module (U, Q, ω) is capable if and only if $Z^*(U, Q, \omega) = 0$.

Originally we defined $Z^*(U, Q, \omega)$ as the intersection of some central crossed submodules of (U, Q, ω), but it coincides with one of the central crossed submodules of the intersection. Consider a projective presentation

$$(V, R, \mu) \rightarrowtail (Y, F, \mu) \xrightarrow{(\pi, \pi')} (U, Q, \omega)$$

of (U, Q, ω). For every normal crossed submodule $(H, X, \mu) \lhd (Y, F, \mu)$ satisfying $[(Y, F, \mu), (V, R, \mu)] \subset (H, X, \mu) \subset (Y, F, \mu)$, we will denote by $(\tilde{H}, \tilde{X}, \tilde{\mu})$ the crossed module

$$\frac{(H, X, \mu)}{[(Y, F, \mu), (V, R, \mu)]}.$$

Let $(p, p') : (Y, F, \mu) \twoheadrightarrow (\tilde{Y}, \tilde{F}, \tilde{\mu})$ denote the canonical projection. Then there is a central extension

$$(\tilde{V}, \tilde{R}, \tilde{\mu}) \rightarrowtail (\tilde{Y}, \tilde{F}, \tilde{\mu}) \xrightarrow{(\tilde{\pi}, \widetilde{\pi'})} (U, Q, \omega)$$

such that $(\tilde{\pi}, \widetilde{\pi'}) \circ (p, p') = (\pi, \pi')$:

$$\begin{array}{ccccc}
[(Y, F, \mu), (V, R, \mu)] & = & [(Y, F, \mu), (V, R, \mu)] & & \\
\downarrow & & \downarrow & & \\
(V, R, \mu) & \rightarrowtail & (Y, F, \mu) & \overset{(\pi, \pi')}{\searrow} & \\
\downarrow & & \downarrow (p, p') & & \\
(\tilde{V}, \tilde{R}, \tilde{\mu}) & \rightarrowtail & (\tilde{Y}, \tilde{F}, \tilde{\mu}) & \xrightarrow[(\tilde{\pi}, \widetilde{\pi'})]{} & (U, Q, \omega).
\end{array}$$

Proposition 11. $Z^*(U, Q, \omega) = (\tilde{\pi}, \widetilde{\pi'})(Z(\tilde{Y}, \tilde{F}, \tilde{\mu}))$.

Proof. We will see that for each central extension

$$(P, N, \partial) \rightarrowtail (T, G, \partial) \xrightarrow{(\varphi, \varphi')} (U, Q, \omega)$$

of (U, Q, ω), there is an inclusion

$$(\tilde{\pi}, \widetilde{\pi'})(Z(\tilde{Y}, \tilde{F}, \tilde{\mu})) \subset (\varphi, \varphi')(Z(T, G, \partial)).$$

Since (Y, F, μ) is projective, there is a morphism (α, α') satisfying

$$(\varphi, \varphi') \circ (\alpha, \alpha') = (\pi, \pi').$$

Then $(\alpha, \alpha')(V, R, \mu) \subset (P, N, \partial)$, $(\alpha, \alpha')[(Y, F, \mu), (V, R, \mu)] = 0$, and (α, α') induces a morphism (β, β') such that $(\alpha, \alpha') = (\beta, \beta') \circ (p, p')$:

$$\begin{array}{ccccc} [(Y,F,\mu),(V,R,\mu)] & \rightarrowtail & (Y,F,\mu) & \xrightarrow{(p,p')} & (\tilde{Y},\tilde{F},\tilde{\mu}) \\ & & & \searrow^{(\alpha,\alpha')} & \downarrow{\scriptstyle (\beta,\beta')} \\ & & & & (T,G,\partial). \end{array}$$

One can easily verify that $(T, G, \partial) = (P, N, \partial) \cdot (\beta, \beta')(\tilde{Y}, \tilde{F}, \tilde{\mu})$, that is, $T = P \cdot \beta(\tilde{Y})$ and $G = N \cdot \beta'(\tilde{F})$.

From this equality we deduce that

$$\begin{aligned}(\beta, \beta')(\mathrm{Z}(\tilde{Y}, \tilde{F}, \tilde{\mu})) &\subset \mathrm{Z}((\beta, \beta')(\tilde{Y}, \tilde{F}, \tilde{\mu})) = (\beta(\tilde{Y})^{\beta'(\tilde{F})}, \mathrm{St}_{\beta'(\tilde{F})}\beta(\tilde{Y}) \cap \mathrm{Z}(\beta'(\tilde{F})), \partial) \\ &\subset (T^{\beta'(\tilde{F})}, \mathrm{St}_G(\beta(\tilde{Y})) \cap C_G(\beta'(\tilde{F})), \partial) = \mathrm{Z}(T, G, \partial)\end{aligned}$$

where $C_G(\beta'(\tilde{F}))$ denotes the centralizer of $\beta'(\tilde{F})$ in G.

Now

$$(\tilde{\pi}, \widetilde{\pi'}) \circ (p, p') = (\pi, \pi') = (\varphi, \varphi') \circ (\alpha, \alpha') = (\varphi, \varphi') \circ (\beta, \beta') \circ (p, p'),$$

which implies $(\tilde{\pi}, \widetilde{\pi'}) = (\varphi, \varphi') \circ (\beta, \beta')$, and then

$$(\tilde{\pi}, \widetilde{\pi'})(\mathrm{Z}(\tilde{Y}, \tilde{F}, \tilde{\mu})) = ((\varphi, \varphi') \circ (\beta, \beta'))(\mathrm{Z}(\tilde{Y}, \tilde{F}, \tilde{\mu})) \subset (\varphi, \varphi')(\mathrm{Z}(T, G, \partial)). \quad \square$$

Example 12. (1) Given a group G, $\mathrm{Z}^*(G, G, \mathrm{Id}) = (\mathrm{Z}^*(G), \mathrm{Z}^*(G), \mathrm{Id})$: for a free presentation $R \rightarrowtail F \overset{\pi}{\twoheadrightarrow} G$ of G,

$$(R, R, \mathrm{Id}) \rightarrowtail (F, F, \mathrm{Id}) \xrightarrow{(\pi,\pi)} (G, G, \mathrm{Id})$$

is a projective presentation of the crossed module (G, G, Id), and therefore

$$(\tilde{\pi}, \tilde{\pi})(\mathrm{Z}(H, H, \mathrm{Id})) = (\tilde{\pi}(\mathrm{Z}(H)), \tilde{\pi}(\mathrm{Z}(H)), \mathrm{Id})$$

where $H = F/[F, R]$. Recall that $\tilde{\pi}(\mathrm{Z}(F/[F, R])) = \mathrm{Z}^*(G)$ by [5, Corollary 3.7, p. 208].

(2) Similarly we deduce that $\mathrm{Z}^*(1, G, i) = (1, \mathrm{Z}^*(G), i)$ from the projective presentation

$$(1, R, i) \rightarrowtail (1, F, i) \xrightarrow{(1,\pi)} (1, G, i)$$

of a crossed module $(1, G, i)$.

(3) For a simply connected crossed module (T, G, ∂), if we denote its precise center $\mathrm{Z}^*(T, G, \partial)$ by $(\mathrm{Z}_1^*, \mathrm{Z}_2^*, \partial)$, then $\mathrm{Z}^*(T) \subset \mathrm{Z}_1^*$ and $\mathrm{Z}^*(G) = \mathrm{Z}_2^*$. Take a free presenta-

tion $V \rightarrowtail F \overset{\pi}{\twoheadrightarrow} T$ of the group T, and $\pi' = \partial \circ \pi$. We obtain a projective presentation

$$(V, R, i) \rightarrowtail (F, F, \mathrm{Id}) \xrightarrow{(\pi,\pi')} (T, G, \partial).$$

Then

$$Z^*(T, G, \partial) = (\tilde{\pi}, \widetilde{\pi'})(Z(H, H, \mathrm{Id})) = (\tilde{\pi}(Z(H)), \widetilde{\pi'}(Z(H)), \partial)$$

where $H = F/[F, R]$, which is also simply connected. But $\widetilde{\pi'}(Z(H)) = Z^*(G)$. Furthermore, since $[F, V] \subset [F, R]$, the canonical induced homomorphisms

$$\tau : \frac{F}{[F, V]} \twoheadrightarrow \frac{F}{[F, R]} \quad \text{and} \quad \varepsilon : \frac{F}{[F, V]} \to T$$

satisfy $\varepsilon = \tilde{\pi} \circ \tau$ and so

$$Z^*(T) = \varepsilon\left(Z\left(\frac{F}{[F, V]}\right)\right) = (\tilde{\pi} \circ \tau)\left(Z\left(\frac{F}{[F, V]}\right)\right) \subset \tilde{\pi} Z\left(\frac{F}{[F, R]}\right).$$

Corollary 13. (U, Q, ω) *is capable if and only if* $Z^*(U, Q, \omega) = 0$.

Proof. If (U, Q, ω) is capable, then trivially there exists a central extension

$$Z(T, G, \partial) \rightarrowtail (T, G, \partial) \xrightarrow{(\varphi,\varphi')} (U, Q, \omega)$$

where $(\varphi, \varphi')(Z(T, G, \partial)) = 0$.

On the other hand, if

$$(V, R, \mu) \rightarrowtail (Y, F, \mu) \xrightarrow{(\pi,\pi')} (U, Q, \omega)$$

is a projective presentation of (U, Q, ω), then

$$(\tilde{\pi}, \widetilde{\pi'})(Z(\tilde{Y}, \tilde{F}, \tilde{\mu})) = Z^*(U, Q, \omega) = 0$$

by Proposition 11. Then $Z(\tilde{Y}, \tilde{F}, \tilde{\mu}) = (\tilde{V}, \tilde{R}, \tilde{\mu})$ and $(U, Q, \omega) \cong I(\tilde{Y}, \tilde{F}, \tilde{\mu})$. □

Example 14. (1) By [5, Proposition 3.9] a group G is capable if and only if $Z^*(G) = 0$. Then G is capable if and only if it is capable as a crossed module in any of the usual ways.

(2) If (T, G, ∂) is a simply connected capable crossed module, then both T and G are capable groups.

Before continuing, we note that it would be interesting to study in depth the connections between unicentrality and capability for crossed modules to the analogous notions from group theory, using the semidirect product of groups. As Gilbert explains in [11], the center of a crossed module (T, G, ∂) is easily obtained from the center $\mathrm{Z}(T \rtimes G) = T^G \rtimes (\mathrm{St}_G(T) \cap \mathrm{Z}(G))$ of the semidirect product $T \rtimes G$. However the equivalence between the notions of centrality for groups and crossed modules does not transfer to an equivalence between the notions of precise center of a group and of a crossed module:

Proposition 15. *Let (U, Q, ω) be a crossed module, and denote the canonical projections from the semidirect product by $p : U \rtimes Q \to U$ and $q : U \rtimes Q \to Q$. Then $(p(\mathrm{Z}^*(U \rtimes Q)), q(\mathrm{Z}^*(U \rtimes Q)), \omega) \subset \mathrm{Z}^*(U, Q, \omega)$.*

Proof. If $(\varphi_1, \varphi_2) : (T, G, \partial) \twoheadrightarrow (U, Q, \omega)$ is a central extension of crossed modules then the homomorphism of groups

$$\varphi_1 \rtimes \varphi_2 : T \rtimes G \twoheadrightarrow U \rtimes Q, \quad (t, g) \mapsto (\varphi_1(t), \varphi_2(g))$$

is clearly a central extension of groups, since $\mathrm{Z}(T \rtimes G) = T^G \rtimes (\mathrm{St}_G(T) \cap \mathrm{Z}(G))$. Then $\mathrm{Z}^*(U, Q, \omega)$ contains the intersection of all those crossed submodules of (U, Q, ω) of the form $(p(\varphi(\mathrm{Z}(E))), q(\varphi(\mathrm{Z}(E))), \omega)$, with $\varphi : E \twoheadrightarrow U \rtimes Q$ a central extension of groups. □

Corollary 16. *Let (U, Q, ω) be a crossed module.*

(1) *If $U \rtimes Q$ is a unicentral group, then (U, Q, ω) is unicentral.*

(2) *If (U, Q, ω) is capable, then $U \rtimes Q$ is a capable group.*

Remark 17. The conditions of this Corollary are sufficient but not necessary, and in general the inclusion in Proposition 15 is not an equality. For example the crossed module $(\mathbb{Z}_2, \mathbb{Z}_2, \mathrm{Id})$ is a unicentral but not a capable crossed module, since $\mathbb{Z}_2$ is a unicentral non-capable group. However the product $\mathbb{Z}_2 \rtimes \mathbb{Z}_2 = \mathbb{Z}_2 \times \mathbb{Z}_2$ is a capable group which is not unicentral.

5 Connections between the precise center and the Ganea map

Theorem 18. *Let (P, N, ∂) be a central crossed submodule of (T, G, ∂).*
Then (P, N, ∂) is contained in $\mathrm{Z}^*(T, G, \partial)$ if and only if, the Ganea map

$$\chi_{(P,N,\partial)} : (P, N, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} \to H_2(T, G, \partial)$$

induced by the extension

$$(P, N, \partial) \rightarrowtail (T, G, \partial) \twoheadrightarrow \frac{(T, G, \partial)}{(P, N, \partial)}$$

is the zero map.

Proof. If we take projective presentations of (T, G, ∂) and $(T, G, \partial)/(P, N, \partial)$ as in Remark 5

$$\begin{array}{ccccc}
 & & (W, S, \mu) & \twoheadrightarrow & (P, N, \partial) \\
 & \nearrow & \downarrow & & \downarrow \\
(V, R, \mu) & \rightarrowtail & (Y, F, \mu) & \xrightarrow{(\pi, \pi')} & (T, G, \partial) \\
 & & & \searrow & \downarrow \\
 & & & & \dfrac{(T, G, \partial)}{(P, N, \partial)}
\end{array}$$

then we know that $\mathrm{Im}(\chi_{(P,N,\partial)}) = [(\tilde{Y}, \tilde{F}, \tilde{\mu}), (\tilde{W}, \tilde{S}, \tilde{\mu})]$. Using Proposition 11 we get a commutative diagram

$$\begin{array}{ccccc}
 & & \mathrm{Z}(\tilde{Y}, \tilde{F}, \tilde{\mu}) & \twoheadrightarrow & \mathrm{Z}^*(T, G, \partial) \\
 & \nearrow & \downarrow & & \downarrow \\
(\tilde{V}, \tilde{R}, \tilde{\mu}) & \rightarrowtail & (\tilde{Y}, \tilde{F}, \tilde{\mu}) & \xrightarrow{(\tilde{\pi}, \widetilde{\pi'})} & (T, G, \partial) \\
 & & {\scriptstyle (\varepsilon, \varepsilon')}\downarrow & & \downarrow{\scriptstyle (\gamma, \gamma')} \\
 & & \mathrm{I}(\tilde{Y}, \tilde{F}, \tilde{\mu}) & \xrightarrow{\cong} & \dfrac{(T, G, \partial)}{\mathrm{Z}^*(T, G, \partial)}.
\end{array}$$

Then we have the following chain of implications:

$$\begin{aligned}
\chi_{(P,N,\partial)} = 0 &\Leftrightarrow [(\tilde{Y}, \tilde{F}, \tilde{\mu}), (\tilde{W}, \tilde{S}, \tilde{\mu})] = 0 \Leftrightarrow (\tilde{W}, \tilde{S}, \tilde{\mu}) \subset \mathrm{Z}(\tilde{Y}, \tilde{F}, \tilde{\mu}) \\
&\Leftrightarrow (\varepsilon, \varepsilon')(\tilde{W}, \tilde{S}, \tilde{\mu}) = 0 \Leftrightarrow ((\gamma, \gamma') \circ (\tilde{\pi}, \widetilde{\pi'}))(\tilde{W}, \tilde{S}, \tilde{\mu}) = 0 \\
&\Leftrightarrow (\tilde{\pi}, \widetilde{\pi'})(\tilde{W}, \tilde{S}, \tilde{\mu}) \subset \mathrm{Z}^*(T, G, \partial).
\end{aligned}$$

Finally $(\tilde{\pi}, \widetilde{\pi'})(\tilde{W}, \tilde{S}, \tilde{\mu}) = (\pi, \pi')(W, S, \mu) = (P, N, \partial)$. □

Remark 19. In this proof it was also shown that for every crossed module (T, G, ∂), the quotient $(T, G, \partial)/\mathrm{Z}^*(T, G, \partial)$ is a capable crossed module since it is isomorphic to the inner actor of another crossed module. In fact, it is possible to prove that $\mathrm{Z}^*(T, G, \partial)$ is the intersection

$$\mathrm{Z}^*(T, G, \partial) = \bigcap \{(P, N, \partial) \lhd (T, G, \partial) \mid (T, G, \partial)/(P, N, \partial) \text{ is capable}\},$$

and so this property gives us another way to define the precise center of a crossed module analogous to the definition of the precise center of a group given in [17].

Next we will construct a HOM functor right adjoint to the tensor product functor, which will help us to give a detailed description of the precise center of a crossed module.

For abelian crossed modules (C, D, g) and (E, F, h) we will denote the following homomorphism of abelian groups by $\mathrm{HOM}((C, D, g), (E, F, h))$:

$$\mathrm{Hom}_{\mathscr{G}rp}(D, E) \to \mathrm{Hom}_{\mathscr{ACM}}((C, D, g), (E, F, h)), \quad r \mapsto (rg, hr).$$

Proposition 20 ([16]). *The functor* $\mathrm{HOM}((C, D, g), -) : \mathscr{ACM} \to \mathscr{ACM}$ *is right adjoint to the functor* $- \otimes (C, D, g)$*, for each abelian crossed module* (C, D, g).

$$\begin{array}{c} \mathscr{ACM} \\ {\scriptstyle -\otimes(C,D,g)}\big\downarrow\big\uparrow{\scriptstyle \mathrm{HOM}((C,D,g),-)} \\ \mathscr{ACM} \end{array}$$

Therefore, for every pair of abelian crossed modules (A, B, f) and (E, F, h) there exists a natural isomorphism of groups

$$\begin{aligned} &\mathrm{Hom}_{\mathscr{ACM}}((A, B, f) \otimes (C, D, g), (E, F, h)) \\ &\quad \cong \mathrm{Hom}_{\mathscr{ACM}}((A, B, f), \mathrm{HOM}((C, D, g), (E, F, h))) \end{aligned}$$

which sends a morphism $(\overline{\langle \phi_1, \phi_2 \rangle}, \psi)$ of abelian crossed modules

$$\begin{array}{ccc} \mathrm{Coker}(\alpha) & \xrightarrow{\bar{\delta}} & B \otimes D \\ {\scriptstyle \overline{\langle \phi_1, \phi_2 \rangle}}\downarrow & & \downarrow{\scriptstyle \psi} \\ E & \xrightarrow{h} & F \end{array}$$

to the morphism (Φ, Ψ)

$$\begin{array}{ccc} A & \xrightarrow{f} & B \\ {\scriptstyle \Phi}\downarrow & & \downarrow{\scriptstyle \Psi} \\ \mathrm{Hom}_{\mathscr{G}rp}(D, E) & \longrightarrow & \mathrm{Hom}_{\mathscr{ACM}}((C, D, g), (E, F, h)) \end{array}$$

defined as follows: for $a \in A$, $b \in B$, $c \in C$ and $d \in D$, $\Phi(a) \in \mathrm{Hom}_{\mathscr{G}rp}(D, E)$ with $\Phi(a)(d) = \phi_2(a \otimes d)$, and $\Psi(b) = (\phi_1^b, \psi^b) \in \mathrm{Hom}_{\mathscr{ACM}}((C, D, g), (E, F, h))$, where $\phi_1^b(c) = \phi_1(b \otimes c)$ and $\psi^b(d) = \psi(b \otimes d)$.

Corollary 21. *For each central crossed submodule* (P, N, ∂) *of* (T, G, ∂) *denote by* $\eta_{(P,N,\partial)}$ *the natural isomorphism of abelian groups*

$$\mathrm{Hom}_{\mathscr{ACM}}((P,N,\partial)\otimes(T,G,\partial)_{\mathrm{ab}},H_2(T,G,\partial))$$
$$\to \mathrm{Hom}_{\mathscr{ACM}}((P,N,\partial),\mathrm{HOM}((T,G,\partial)_{\mathrm{ab}},H_2(T,G,\partial))).$$

Then $\mathrm{Z}^*(T,G,\partial)=\mathrm{Ker}(\eta_{\mathrm{Z}(T,G,\partial)}(\chi_{\mathrm{Z}(T,G,\partial)}))$.

Proof. We have $(P,N,\partial)\subset \mathrm{Z}^*(T,G,\partial)$ if and only if $\chi_{(P,N,\partial)}=0$, and this holds if and only if $\eta_{(P,N,\partial)}(\chi_{(P,N,\partial)})=0$, by Theorem 18. But

$$\eta_{(P,N,\partial)}(\chi_{(P,N,\partial)})=\eta_{(P,N,\partial)}(\chi_{\mathrm{Z}(T,G,\partial)}\circ(i\otimes(T,G,\partial)_{\mathrm{ab}}))=\eta_{\mathrm{Z}(T,G,\partial)}(\chi_{\mathrm{Z}(T,G,\partial)})\circ i,$$

where i denotes the inclusion $(P,N,\partial)\hookrightarrow \mathrm{Z}^*(T,G,\partial)$. □

If we compute the kernel of $\eta_{\mathrm{Z}(T,G,\partial)}(\chi_{\mathrm{Z}(T,G,\partial)})$, we get the following characterization of the precise center:

Corollary 22. *If we denote* $\mathrm{Z}^*(T,G,\partial)$ *by* $(\mathrm{Z}_1^*,\mathrm{Z}_2^*,\partial)$, *then*

$$\mathrm{Z}_1^*=\{r\in T^G\,|\,\gamma_2(r\otimes\bar{g})=0 \textit{ for every } \bar{g}\in G_{\mathrm{ab}}\},$$

and

$$\mathrm{Z}_2^*=\left\{m\in \mathrm{St}_G(T)\cap \mathrm{Z}(G)\,\middle|\,\begin{array}{l}\gamma_1(m\otimes\bar{t})=0 \textit{ for all } \bar{t}\in T/[G,T]\\ \gamma'(m\otimes\bar{g})=0 \textit{ for all } \bar{g}\in G_{\mathrm{ab}}\end{array}\right\},$$

where γ_1, γ_2 *and* γ' *are the morphisms defined in Theorem* 4.

Remark 23. It is clear that Z_2^* is contained in

$$\mathrm{Z}^*(G)=\{x\in \mathrm{Z}(G)\,|\,\gamma_G(x\otimes\bar{g})=0 \text{ for every } \bar{g}\in G_{\mathrm{ab}}\},$$

where γ_G denotes the classical Ganea map $\mathrm{Z}(G)\otimes G_{\mathrm{ab}}\to H_2(G)$.

From Theorem 18 we deduce the following characterization of capable crossed modules:

Proposition 24. *A crossed module* (T,G,∂) *is capable if and only if for every non-zero* $y\in \mathrm{St}_G(T)\cap \mathrm{Z}(G)$ *and non-zero* $x\in \mathrm{Ker}(\partial)\cap T^G$, *the Ganea maps associated to the central crossed submodules* $(0,\langle y\rangle,\partial)$ *and* $(\langle x\rangle,0,\partial)$ *are non-zero maps.*

Proof. It suffices to prove that if (T,G,∂) is not capable, then there exists some element y or x satisfying the conditions of the proposition such that $(\langle x\rangle,0,\partial)$ or $(0,\langle y\rangle,\partial)$ is in $\mathrm{Z}^*(T,G,\partial)$.

If (T,G,∂) is not capable then $\mathrm{Z}^*(T,G,\partial)=(\mathrm{Z}_1^*,\mathrm{Z}_2^*,\partial)\neq 0$, and there will be a non-zero element $x\in R\subset T^G$ or a non-zero element $y\in M\subset \mathrm{St}_G(T)\cap \mathrm{Z}(G)$. In this latter case, $(0, \langle y \rangle, \partial) \subset Z^*(T, G, \partial)$. In the first case, for $0 \neq x \in R \subset T^G$, if $\partial(x) \neq 0$ then $(0, \langle \partial(x) \rangle, \partial) \subset Z^*(T, G, \partial)$, and if $x \in \mathrm{Ker}(\partial)$ then

$$(\langle x \rangle, 0, \partial) \subset Z^*(T, G, \partial). \quad \square$$

Remark 25. In this result, $\mathrm{St}_G(T) \cap Z(G)$ can be easily replaced by $\mathrm{St}_G(T) \cap Z^*(G)$.

Corollary 26. *If (T, G, ∂) is a capable simply connected crossed module, then it is aspherical and $(T, G, \partial) \cong (G, G, \mathrm{Id})$ with G a capable group.*

Proof. For every non-zero $x \in \mathrm{Ker}(\partial) \cap T^G = \mathrm{Ker}(\partial) \cap Z(T) = \mathrm{Ker}(\partial)$ the Ganea map associated to the crossed submodule $(\langle x \rangle, 0, \partial)$ is non-zero. But

$$(\langle x \rangle, 0, \partial) \otimes (T/[G, T]), G_{\mathrm{ab}}, \bar{\partial}) = 0$$

since $\bar{\partial}$ is surjective, and so $\mathrm{Ker}(\partial) = 0$. The corollary follows from Example 14. $\square$

Example 27. (1) An example of a non-aspherical capable crossed module is the quotient $(A, G, 0)/Z(A, G, 0)$ with A a non-trivial G-module, since the center of $(A, G, 0)$ is $(A^G, \mathrm{St}_G(A) \cap Z(G), 0)$ and $A/A^G \neq 0$.

(2) Using the previous example we can get a capable crossed module (T, G, ∂) with T and G non-capable groups. Consider the action of $\mathbb{Z}_2$ on $\mathbb{Z}_2 \oplus \mathbb{Z}_2$ given by ${}^1(1, 0) = (0, 1)$ and ${}^1(0, 1) = (1, 0)$, and then our capable crossed module is

$$\frac{(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)}{Z(\mathbb{Z}_2 \oplus \mathbb{Z}_2, \mathbb{Z}_2, 0)} = (\mathbb{Z}_2, \mathbb{Z}_2, 0),$$

but $\mathbb{Z}_2$ is a non-capable group; see [5]. This is an example of a crossed module whose precise center $Z^*(T, G, \partial) = (Z_1^*, Z_2^*, \partial)$ satisfies the inclusions $Z_1^* \not\subseteq Z^*(T)$ and $Z_2^* \not\subseteq Z^*(G)$.

(3) The abelian crossed module $(T, G, \partial) = (\mathbb{Z}_4 \times \mathbb{Z}_4, \mathbb{Z}_4 \times \mathbb{Z}_2, \mathrm{Id} \times p)$, where $p : \mathbb{Z}_4 \twoheadrightarrow \mathbb{Z}_2$ denotes the canonical projection, is an example of a simply connected crossed module whose precise center $Z^*(T, G, \partial) = (Z_1^*, Z_2^*, \partial)$ satisfies the inclusion $Z^*(T) \not\subseteq Z_1^*$. $Z^*(T) = 0$ since $\mathbb{Z}_4 \times \mathbb{Z}_4$ is a capable group (see [5]), but $Z_1^* \neq 0$ since (T, G, ∂) is not capable, by Corollary 26.

From Theorem 18 we can also deduce the following characterization of the unicentral crossed modules:

Corollary 28. *For a crossed module (T, G, ∂) are equivalent:*

(1) *(T, G, ∂) is unicentral;*

(2) *the Ganea map $\chi_{Z(T,G,\partial)} : Z(T, G, \partial) \otimes (T, G, \partial)_{\mathrm{ab}} \to H_2(T, G, \partial)$ is zero;*

(3) *the canonical homomorphism*

$$H_2(T,G,\partial) \to H_2\left(\frac{(T,G,\partial)}{Z(T,G,\partial)}\right)$$

is injective.

Example 29. (1) Since $H_2(T,G,\partial)=0$, every projective crossed module is unicentral.

(2) A group G is unicentral if and only if it is unicentral considered as a crossed module in any of the usual ways.

(3) If T is a unicentral group, then every simply connected crossed module (T,G,∂) is unicentral. Write $Z^*(T,G,\partial)=(Z_1^*,Z_2^*,\partial)$; then

$$Z(T)=Z^*(T)\subset Z_1^*\subset Z(T) \quad \text{and} \quad \partial(Z(T))=\partial(Z_1^*)\subset Z_2^*\subset \partial(Z(T)),$$

and so

$$Z^*(T,G,\partial)=(Z(T),\partial(Z(T)),\partial)=Z(T,G,\partial).$$

(4) If (T,G,∂) is a unicentral crossed module satisfying $Z(G)\subset \mathrm{St}_G(T)$, then G must clearly be unicentral. This property can be applied to the crossed modules of the form (N,G,i) with $N \lhd G$ or $(A,G,0)$ with A a trivial G-module.

Corollary 30. *Let (T,G,∂) be a simply connected crossed module. If G is a unicentral group then (T,G,∂) is unicentral.*

We will need the following lemma to prove Corollary 30:

Lemma 31. *If (T,G,∂) is a simply connected crossed module then $H_2(T,G,\partial)$ is aspherical.*

Proof. Take a projective presentation $(V,R,i)\rightarrowtail(F,F,\mathrm{Id})\twoheadrightarrow(T,G,\partial)$ with i injective (see Example 12); then

$$H_2(T,G,\partial)\cong\left(\frac{V\cap[F,F]}{[F,R]},\frac{R\cap[F,F]}{[F,R]},\bar{i}\right)$$

is aspherical. □

Proof of Corollary 30. Denote by (σ,σ') the morphism

$$H_2(T,G,\partial) \to H_2\left(\frac{(T,G,\partial)}{Z(T,G,\partial)}\right).$$

By Lemma 31, $H_2(T, G, \partial)$ is aspherical, so we only have to prove that

$$\sigma' : H_2(G) \to H_2\left(\frac{G}{\partial(\mathrm{Z}(T))}\right)$$

is injective. But $H_2(G) \to H_2(G/\mathrm{Z}(G))$ is injective since G is unicentral, and then σ' is injective too. □

6 Connections to relatively capable groups and capable pairs of groups

In [17], Shahriari began the study of the normal structure of capable groups, by showing that certain groups, called *relatively capable* groups, can be normal subgroups of capable groups. Ellis continued in [9] the research on relatively capable groups, and proposed an extension of capability theory for groups to a theory for pairs of groups. By a pair of groups (G, N) he understands a group G and a normal subgroup N. A *capable pair* is a pair of groups (G, N) such that there exists a group M and a crossed module $\partial : M \to G$ satisfying $\partial(M) = N$ and $\mathrm{Ker}(\partial) = M^G$.

In a capable pair (G, N) the group G is not necessarily capable. For example, $(G, 1)$ is clearly a capable pair for every group G.

The connection between the two notions is given by the following proposition:

Proposition 32 ([9, Proposition 2]). *A group N is relatively capable if and only if it is a normal subgroup of some group G for which the pair (G, N) is capable.*

In the next proposition we show that both of the notions are related to our notion of capable crossed module when we consider the normal subgroup $N \lhd G$ as the inclusion crossed module (N, G, i).

Proposition 33. *Let N be a normal subgroup of a group G.*

(1) *If G is a capable group, then (N, G, i) is a capable crossed module.*

(2) *If (N, G, i) is a capable crossed module, then (G, N) is a capable pair of groups.*

Proof. (1) If we write $\mathrm{Z}^*(N, G, i) = (\mathrm{Z}_1^*, \mathrm{Z}_2^*, i)$, then, by Remark 23,

$$\mathrm{Z}_1^* \subset \mathrm{Z}_2^* \subset \mathrm{Z}^*(G) = 0$$

and (N, G, i) is capable.

(2) Since (N, G, i) is capable there exists a crossed module (M, R, ∂) making the following diagram commutative:

$$\begin{array}{ccccc}
M^R & \rightarrowtail & M & \overset{\varphi}{\twoheadrightarrow} & N \\
\downarrow & & \downarrow{\scriptstyle \partial} & & \downarrow{\scriptstyle i} \\
\mathrm{St}_R(M) \cap \mathrm{Z}(R) & \rightarrowtail & R & \overset{\varphi'}{\twoheadrightarrow} & G.
\end{array}$$

The action of R on M induces an action of G on M. With this action $\varphi'\partial$ is a crossed module satisfying

$$\mathrm{Im}(\varphi'\partial) = \mathrm{Im}(i\varphi) = N \quad \text{and} \quad \mathrm{Ker}(\varphi'\partial) = \mathrm{Ker}(i\varphi) = \mathrm{Ker}(\varphi) = M^R = M^G,$$

and then the pair (G, N) is capable. □

The converse of Proposition 33 (2) is not true. For example, if G is a non-capable group then $(G, 1)$ is a capable pair, while $(1, G, i)$ is not a capable crossed module. However we have the following Corollary:

Corollary 34. *A group N is relatively capable if and only if it is a normal subgroup of some group G for which the crossed module (N, G, i) is capable.*

7 Applications to perfect crossed modules and universal central extensions

Definition 5. A crossed module (T, G, ∂) is said to be *perfect* if it coincides with its commutator crossed submodule

$$(T, G, \partial) = [(T, G, \partial), (T, G, \partial)].$$

This means that G is a perfect group and $T = [G, T]$.

Remark 35. By Corollary 28, every perfect crossed module (T, G, ∂) is unicentral since $(T, G, \partial)_{\mathrm{ab}} = 0$ and $\chi_{Z(T,G,\partial)} = 0$.

Algebraic K-theory provides important examples of unicentral crossed modules:

Example 36. (1) Given a two-sided ideal I of a ring R, we can construct the perfect crossed modules $(E(I), E(R), i)$, $(\mathrm{St}(R, I), St(R), \bar{\gamma})$ and $(K_2(R, I), K_2(R), \bar{\gamma})$, where $E(I)$ and $E(R)$ are the groups of elementary matrices with coefficients in I and R, $\mathrm{St}(R)$ and $K_2(R)$ denote the Steinberg group and the second K-theory group of the ring R, $\mathrm{St}(R, I)$ is the relative Steinberg group defined by Keune [12] and $K_2(R, I)$ denotes the second relative K-theory group introduced by Loday [14] and Keune [12]. By Corollary 28, all of these are unicentral crossed modules.

(2) For a ring R, consider the groups $M_n(A)$ of $n \times n$ matrices with coefficients in an R-bimodule A. Let $M(A)$ be the inductive limit of the groups $M_n(A)$, and denote by $M_0(A)$ the $E(R)$-module generated by the matrices $E_{ij}(a) \in M(A)$ with exactly one non-zero entry $a \in A$ in the (i, j) position, with $i \neq j$. Then $(M_0(A), E(R), 0)$ is a perfect and so unicentral crossed module.

Dennis and Igusa constructed in [7] an additive Steinberg $\mathrm{St}(R)$-module $\mathrm{St}(R, A)$ and a relative K-theory group $K_2(R, A)$ such that the crossed modules $(\mathrm{St}(R, A), \mathrm{St}(R), 0)$ and $(K_2(R, A), K_2(R), 0)$ are also perfect, and thus unicentral.

It is known that for groups the following is true [5, p. 117]:

Proposition 37. *If G is a perfect group, then* $Z(G/Z(G)) = 1$.

The analogue of this proposition for crossed modules is also true:

Proposition 38. *If* (T, G, ∂) *is a perfect crossed module, then*

$$Z((T, G, \partial)/Z(T, G, \partial)) = 0.$$

Proof. Clearly $(T, G, \partial)/Z(T, G, \partial)$ is a perfect crossed module, and by Remark 35 it is unicentral. Then

$$Z\left(\frac{(T, G, \partial)}{Z(T, G, \partial)}\right) = (\varphi, \varphi')(Z(T, G, \partial)) = 0,$$

where (φ, φ') is the central extension

$$Z(T, G, \partial) \rightarrowtail (T, G, \partial) \xrightarrow{(\varphi, \varphi')} \frac{(T, G, \partial)}{Z(T, G, \partial)}. \qquad \square$$

In [15], Norrie defined for any crossed module (T, G, ∂) the group

$$\tau_G(T) = \{t \in T \mid {}^g t t^{-1} \in T^G \text{ for all } g \in G\}.$$

Clearly $T^G \subset \tau_G(T) \subset T$, and (T, G, ∂) is called *fixed-point constrained* if $\tau_G(T) = T^G$. Norrie does not prove Proposition 38, but the following:

Proposition 39. *If* (T, G, ∂) *is a perfect crossed module then*

$$Z((T, G, \partial)/Z(T, G, \partial)) = 0$$

if and only if (T, G, ∂) *is fixed-point constrained.*

From Propositions 38 and 39 we conclude the following result conjectured by Norrie in [15, p. 86]:

Corollary 40. *If* (T, G, ∂) *is a perfect crossed module then it is fixed-point constrained.*

From this Corollary it is now easy to deduce the following result which Norrie tried to prove in [15, pp. 115–126]:

Corollary 41. *Let* $\beta = (\beta_1, \beta_2) : (U_1, U_2, \delta) \twoheadrightarrow (T, G, \partial)$ *be a crossed module central extension of a crossed module* (T, G, ∂). *Then* β *is the crossed module universal central extension of* (T, G, ∂) *if and only if* (U_1, U_2, δ) *is a perfect crossed module and every crossed module central extension of* (U_1, U_2, δ) *splits.*

Proof. The sufficient condition is [15, Theorem 2.60]. The necessary condition follows from [15, Theorem 2.64] and Corollary 40. □

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