Universal Central Extensions of Precrossed Modules and Milnor's relative K_2

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Abstract

We characterize the universal central extension of a perfect precrossed module giving two descriptions, one in terms of non abelian tensor products of groups and other in terms of projective presentations. As application to relative algebraic K-theory, we obtain that Milnor's absolute and relative K_2 groups are the kernel of the universal central extension of the precrossed module determined by the groups of the elementary matrices of a ring and relative to an ideal, respectively.

Key words: universal central extension, precrossed module, Steinberg group, relative K-theory, non abelian tensor product
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1 Introduction

In [1] we proved that the category of precrossed modules is an algebraic category and we defined cotriple homology and cohomology theories of precrossed modules. These theories were shown to generalize the Eilenberg-MacLane (co)homology groups if we consider a group G as a precrossed module $(1, G, i)$ or $(G, 1, 1)$. They also extend the low dimensional homology for crossed modules of Gilbert [9]. These theories are different from the homology groups of a precrossed P-module, where P is a fixed action group, defined in [7] and [10].

The aim of this article is to characterize the universal central extension of a perfect precrossed module. It is known that for a group P there exists the universal central extension of P if and only if P is a perfect group, and it is

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given by a commutator map $P \otimes P \twoheadrightarrow P$ with kernel $H_2(P)$ [5]. To generalize this result we introduce the notion of *perfect* precrossed module, that is, a precrossed module which coincides with its commutator precrossed submodule, and we prove that a precrossed module (M, P, μ) admits a universal central extension if and only if (M, P, μ) is a perfect precrossed module.

We construct the universal central extension of a perfect precrossed module (M, P, μ) in terms of a projective presentation of (M, P, μ). Another description of the universal central extension, in terms of non abelian tensor products of groups, is given. We obtain that the universal central extension of (M, P, μ) takes the form

$$H_2(M, P, \mu) \rightarrowtail (M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu) \twoheadrightarrow (M, P, \mu)$$

where $H_2(M, P, \mu)$ denotes the second homology of the precrossed module (M, P, μ). So the universal central extension of a precrossed module generalizes the universal central extension of a group.

There are several articles in the literature concerning the universal central extension in the category of crossed modules [15,16,9,6,17]. To emphasize the difference between the universal central extensions in the categories of precrossed and crossed modules we describe an example of a universal precrossed (not crossed) central extension. We make use of some tools from Algebraic K-theory like the Milnor's relative K-theory group $K_2(I)$ associated to a two-sided ideal I of a ring R. There is a perfect crossed module $(E(I), E(R), i)$ formed by the group $E(R)$ of the elementary matrices of R, and the group of elementary matrices relative to the ideal I. The universal precrossed central extension of $(E(I), E(R), i)$ is

$$(K_2(I), K_2(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i)$$

where $St(R)$ and $K_2(R)$ denote the Steinberg group and the second K-theory group of the ring R, and $St(I)$ and $K_2(I)$ denote their Stein relativizations [18]. As a consequence we get the isomorphisms of precrossed modules $H_2(E(I), E(R), i) \cong (K_2(I), K_2(R), \gamma)$ and $(St(I), St(R), \gamma) \cong (E(I) \otimes (E(I) \rtimes E(R)), E(R) \otimes E(R), i \otimes \nu)$ which provide expressions for computing the relative groups $St(I)$ and $K_2(I)$. Next, we show that the quotient by the Peiffer elements of the above central extension is the universal crossed central extension showed in [9]

$$(St(R, I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i)$$

where $St(R, I)$ denotes the relative Steinberg group introduced by Loday [13] and Keune [11]. An important consequence is that the kernel of $(St(R, I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i)$ established in [9] is not the correct one. The correct one is the second homology of crossed modules [6] of Carrasco, Cegarra and Grandjeán $H_2^{CCG}(E(I), E(R), i)$, or equivalently the second homology of crossed modules [16] of Grandjeán and Ladra $H_2^{GL}(E(I), E(R), i)$.

We begin in section 2 by recalling standard facts about the category of precrossed modules such as its tripleability and the existence and construction of free precrossed modules. Then, we make a short description of the precrossed module homology developed in [1] completed with a five term exact sequence in homology of precrossed modules and a Hopf type formula for the second homology of a precrossed module. This five term exact sequence in homology generalizes the Stallings and Stammbach five term exact sequence in integral homology of groups, by the same way the Hopf's formula for the second homology of a precrossed module generalizes the classical Hopf's formula for the second integral homology group.

In section 3 we make the general study of the universal central extensions of precrossed modules. We obtain expressions of the universal central extension of a perfect precrossed module (M, P, μ) in terms of a fixed projective presentation of (M, P, μ), and in terms of the non abelian tensor product of groups.

Finally, in section 4 we work out the application to relative algebraic K-theory described above.

2 Homology of precrossed modules

A *precrossed module* (M, P, μ) is a group homomorphism $\mu : M \to P$ together with an action of P on M, denoted ${}^{p}m$ for $p \in P$ and $m \in M$, satisfying $\mu({}^{p}m) = p\mu(m)p^{-1}$ for all $p \in P$ and $m \in M$. If in addition it verifies the Peiffer's identity ${}^{\mu(m)}m' = mm'm^{-1}$ for all $m, m' \in M$, we say that (M, P, μ) is a *crossed module.*

Example 1 *Let P, G be groups with P acting on G non trivially or with G a non abelian group. Then $(G \rtimes P, P, \pi)$ is a precrossed module and not a crossed module, where $G \rtimes P$ denotes the semidirect product of G and P, π is the natural surjection from $G \rtimes P$ onto P and the action of P on $G \rtimes P$ is ${}^{p'}(g, p) = ({}^{p'}g, {}^{p'}p)$ [4]. In particular, if G is a non abelian group, $(G, 1, 1)$ is a precrossed module and is not a crossed module.*

A *precrossed module morphism* $(\Phi, \Psi) : (M_1, P_1, \mu_1) \longrightarrow (M_2, P_2, \mu_2)$ is a pair of group homomorphisms $\Phi : M_1 \to M_2$ and $\Psi : P_1 \to P_2$ such that $\Psi \circ \mu_1 = \mu_2 \circ \Phi$ and $\Phi({}^{p}m) = {}^{\Psi(p)}\Phi(m)$ for all $p \in P_1$ and $m \in M_1$.

We denote the category of precrossed modules by $\mathcal{PCM}$.

A morphism (Φ, Ψ) in $\mathcal{PCM}$ is said to be *injective (surjective)* if both Φ and Ψ are injective (surjective) group homomorphisms.

A *precrossed submodule* (N, Q, μ') of a precrossed module (M, P, μ) is a precrossed module such that N and Q are respectively subgroups of M and P, the action of Q on N is induced by the one of P on M and $\mu_{|N} = \mu'$. It is said to be a *normal precrossed submodule* if besides N and Q are normal in M and P, ${}^{p}n \in N$ and ${}^{q}mm^{-1} \in N$ for all $p \in P$, $q \in Q$, $m \in M$ and $n \in N$.

If (N, Q, μ) is a normal precrossed submodule of (M, P, μ), we define the *quotient precrossed module* $(M, P, \mu)/(N, Q, \mu)$ as $(M/N, P/Q, \overline{\mu})$ where the homomorphism $\overline{\mu}$ is induced by μ and P/Q acts on M/N by ${}^{pQ}mN = ({}^{p}m)N$ for $p \in P$ and $m \in M$.

We call *Peiffer subgroup* of a precrossed module (M, P, μ) the subgroup $\langle M, M\rangle$ of M generated by the Peiffer elements $m_1 m_2 m_1^{-1\mu(m_1)} m_2^{-1}$ with $m_1, m_2 \in M$. It is a normal subgroup of M, and the quotient $(M, P, \mu)/(\langle M, M\rangle, 1, 1)$ is a crossed module.

The *kernel* of a precrossed module morphism $(\Phi, \Psi) : (M_1, P_1, \mu_1) \to (M_2, P_2, \mu_2)$ is the normal precrossed submodule $(Ker\Phi, Ker\Psi, \mu_1)$ of (M_1, P_1, μ_1). Its *image* is the precrossed submodule $(Im\Phi, Im\Psi, \mu_2)$ of (M_2, P_2, μ_2).

In [1] we introduced analogues to some basic concepts from group theory, like centre or commutator subgroups, in the category of precrossed modules. In the case of crossed modules these concepts were introduced by Norrie [15].

The *centre* $Z(M, P, \mu)$ of a precrossed module (M, P, μ) is the normal precrossed submodule $(Inv(M) \cap Z(M), St_P(M) \cap Z(P), \mu)$, where $St_P(M)$ denotes the group $\{p \in P \mid {}^{p} m = m \text{ for all } m \in M\}$, $Inv(M) = \{m \in M \mid \mu(m) \in St_P(M) \text{ and } {}^{p}m = m \text{ for all } p \in P\}$ and $Z(M)$, $Z(P)$ denote the centres of M and P. $Z(M, P, \mu)$ is the maximal central precrossed submodule of (M, P, μ).

A precrossed module (M, P, μ) is said to be *abelian* if $(M, P, \mu) = Z(M, P, \mu)$. Equivalently M and P are abelian groups and P acts trivially on M.

If (N, Q, μ) and (R, K, μ) are normal precrossed submodules of (M, P, μ), we define the *commutator precrossed submodule* $[(N, Q, \mu), (R, K, \mu)]$ *of* (N, Q, μ) *and* (R, K, μ) as the normal precrossed submodule $([Q, R]\,[K, N]\,[N, R]\,, [Q, K]\,, \mu)$ of (M, P, μ), where $[Q, R]$ denotes the normal subgroup of M generated by the elements $\{{}^{q}rr^{-1} \mid q \in Q, r \in R\}$, $[K, N]$ denotes the normal subgroup of M generated by the elements $\left\{{}^{k}nn^{-1} \mid k \in K, n \in N\right\}$ and $[N, R]$ and $[Q, K]$ denote the usual commutator subgroups of N with R and Q with K.

In particular, the *commutator precrossed submodule* of a precrossed module (M, P, μ) is $[(M, P, \mu), (M, P, \mu)] = ([M, M]\,[P, M], \,[P, P]\,, \mu)$. It is the smallest normal precrossed submodule of (M, P, μ) making the quotient an abelian precrossed module.

The inclusion of abelian precrossed modules $\mathcal{APCM}$ in $\mathcal{PCM}$ has a left adjoint $ab : \mathcal{PCM} \longrightarrow \mathcal{APCM}$ termed the *abelianisation functor*, which assigns to a precrossed module (M, P, μ) the abelian precrossed module $(M, P, \mu)_{ab} = (M/[M, M]\,[P, M]\,,\ P/\,[P, P]\,, \overline{\mu})$.

The forgetful functor $\mathcal{U} : \mathcal{PCM} \longrightarrow \mathcal{S}et$, $\mathcal{U}(M, P, \mu) = M \times P$, that assigns to each precrossed module (M, P, μ) the cartesian product of the underlying sets M and P, is tripleable. Its left adjoint, the free precrossed module functor $\mathcal{F} : \mathcal{S}et \longrightarrow \mathcal{PCM}$ is given by $\mathcal{F}(X) = (\overline{F}, F{*}F, \langle i_1, Id\rangle_{|\overline{F}})$, where F is the free group over X, $\overline{F} = Ker(F{*}(F{*}F) \xrightarrow{\langle 0,Id\rangle} F{*}F)$, $\langle i_1, Id\rangle : F{*}(F{*}F) \longrightarrow F{*}F$, $i_1 : F \rightarrowtail F * F$ is the first inclusion in the coproduct, and $F * F$ acts on $\overline{F}$ by conjugation.

The category $\mathcal{PCM}$ has enough projective objects. Each precrossed module admits a presentation by a projective precrossed module through the counit of the adjunction between $\mathcal{U}$ and $\mathcal{F}$. In [1] it can be found an useful construction of a family of projective precrossed modules.

For a precrossed module (M, P, μ) let us consider the cotriple resolution $C.(M, P, \mu) \rightarrow (M, P, \mu)$ associated to the functors $\mathcal{F}$ and $\mathcal{U}$. Particularizing Barr and Beck's cotriple homology [2], we define, for $n \geq 1$, the *homology precrossed modules* of the precrossed module (M, P, μ) by

$$H_n(M, P, \mu) = H_{n-1}((C.(M, P, \mu))_{ab}\,, \partial_*)\ [1].$$

Theorem 2 (Five term exact sequence in homology of precrossed modules) *Let* $0 \longrightarrow (R, K, \partial) \xrightarrow{i} (T, G, \partial) \xrightarrow{p} (M, P, \mu) \longrightarrow 0$ *be an exact sequence of precrossed modules. There exists a natural exact sequence of abelian precrossed modules*

$$\begin{aligned} H_2(T, G, \partial) \longrightarrow H_2(M, P, \mu) &\longrightarrow (R, K, \partial)/\,[(T, G, \partial), (R, K, \partial)] \\ &\longrightarrow H_1(T, G, \partial) \longrightarrow H_1(M, P, \mu) \longrightarrow 0 \end{aligned}$$

The proof of Theorem 2 is analogous to the one of [6, Theorem 12]. It requires the following routine extension of [6, Lemma 11]:

Lemma 3 *Let* $0 \longrightarrow (R, K, \partial) \xrightarrow{i} (T, G, \partial) \xrightarrow{p} (M, P, \mu) \longrightarrow 0$ *be an exact sequence of precrossed modules. If* p *admits a section, then the sequence*

$$0 \rightarrow (R, K, \partial)/\,[(T, G, \partial), (R, K, \partial)] \rightarrow H_1(T, G, \partial) \rightarrow H_1(M, P, \mu) \rightarrow 0$$

is a split short exact sequence of abelian precrossed modules.

Theorem 2 will help us to give a Hopf type formula for the second homology precrossed module $H_2(M,P,\mu)$, in terms of an arbitrary projective presentation of a precrossed module (M,P,μ).

Corollary 4 *If $(V,R,\tau) \rightarrowtail (Y,F,\tau) \overset{\pi}{\twoheadrightarrow} (M,P,\mu)$ is a projective presentation of a precrossed module (M,P,μ), then there exists a natural isomorphism of abelian precrossed modules*

$$H_2(M,P,\mu) \cong (V,R,\tau) \cap [(Y,F,\tau),(Y,F,\tau)] \,/\, [(Y,F,\tau),(V,R,\tau)]$$

Proof Apply the five term exact sequence to the projective presentation of (M,P,μ). Since $H_2(Y,F,\tau)=0$ it follows that
$Ker((V,R,\tau)\diagup[(Y,F,\tau),(V,R,\tau)] \longrightarrow (Y,F,\tau)\diagup[(Y,F,\tau),(Y,F,\tau)]) = (V,R,\tau)\cap[(Y,F,\tau),(Y,F,\tau)]\diagup[(Y,F,\tau),(V,R,\tau)] \cong H_2(M,P,\mu)$. ∎

Example 5 *The analogous classical theorems in integral homology of groups can be deduced from these results. If we take a short exact sequence of groups $0 \to K \to G \to P \to 0$, we can consider it as a sequence of precrossed modules $0 \to (K,1,1) \to (G,1,1) \to (P,1,1) \to 0$ or $0 \to (1,K,i) \to (1,G,i) \to (1,P,i) \to 0$ and the resulting five term exact sequence of Theorem 2 is the five term exact sequence in integral homology of groups*

$$H_2(G) \longrightarrow H_2(P) \longrightarrow K/\,[G,K] \longrightarrow H_1(G) \longrightarrow H_1(P) \longrightarrow 0$$

considered as precrossed modules in the corresponding way.

On the other hand, the Hopf's formula for a projective presentation $(Y,R,\tau) \rightarrowtail (Y,F,\tau) \overset{\pi}{\twoheadrightarrow} (1,P,i)$ provides the Hopf's formula for the group $H_2(P)$ considered as a precrossed module $H_2(1,P,i) \cong (1, R\cap[F,F]\,/\,[F,R]\,,i)$.

3 Universal central extension of a perfect precrossed module

A precrossed module (M,P,μ) is said to be *perfect* if (M,P,μ) coincides with its commutator precrossed submodule. This means that P is a perfect group and $M=[M,M]\,[P,M]$ (in the notation of [13] P is a perfect group and M is a P-perfect P-group). Using the fact that the category of precrossed modules is equivalent to the category of simplicial groups of length 1 [3], observe that a precrossed module is perfect if and only if the corresponding truncated simplicial group is perfect in dimension 1.

We call a *central extension of* (M,P,μ) a surjective morphism of precrossed modules $\psi=(\psi_1,\psi_2):(X_1,X_2,\delta) \twoheadrightarrow (M,P,\mu)$ such that $Ker\psi \subset Z(X_1,X_2,\delta)$.

A central extension $\phi : (U_1, U_2, \omega) \twoheadrightarrow (M, P, \mu)$ of (M, P, μ) is said to be *universal* if for every central extension $\psi : (X_1, X_2, \delta) \twoheadrightarrow (M, P, \mu)$ of (M, P, μ) there exists an unique morphism $f : (U_1, U_2, \omega) \to (X_1, X_2, \delta)$ making commutative the following diagram

$$\begin{array}{ccc} (U_1, U_2, \omega) & \stackrel{\phi}{\twoheadrightarrow} & (M, P, \mu) \\ \downarrow f & & \| \\ (X_1, X_2, \delta) & \stackrel{\psi}{\twoheadrightarrow} & (M, P, \mu) \end{array}$$

Lemma 6 *Let $\alpha = (\alpha_1, \alpha_2) : (Y_1, Y_2, \sigma) \twoheadrightarrow (M, P, \mu)$ and $\beta = (\beta_1, \beta_2) : (X_1, X_2, \delta) \twoheadrightarrow (M, P, \mu)$ be two central extensions of a precrossed module (M, P, μ), with (Y_1, Y_2, σ) a perfect precrossed module.*

Then there exists at most one precrossed module morphism

$$f : (Y_1, Y_2, \sigma) \to (X_1, X_2, \delta)$$

such that $\alpha = \beta f$.

Proof Suppose that there are two morphisms $f = (f_1, f_2)$ and $g = (g_1, g_2)$ making the following diagram commute

$$\begin{array}{ccc} (Y_1, Y_2, \sigma) & \stackrel{\alpha}{\twoheadrightarrow} & (M, P, \mu) \\ f \downarrow \ \downarrow g & & \| \\ (X_1, X_2, \delta) & \stackrel{\beta}{\twoheadrightarrow} & (M, P, \mu) \end{array}$$

For each element $a \in Y_2$ there exists an element $k_a \in Ker\beta_2 \subset Z(X_2) \cap st_{X_2}(X_1)$ such that $f_2(a) = g_2(a)k_a$ since $g_2(a)^{-1}f_2(a) \in Ker\beta_2$. Analogously for each element $b \in Y_1$ there exists an element $c_b \in Ker\beta_1 \subset Z(X_1) \cap Inv(X_1)$ such that $f_1(b) = g_1(b)c_b$.

We will check that f_2 and g_2 coincide in the commutators of elements of Y_2: for each $z, y \in Y_2$, $f_2(zyz^{-1}y^{-1}) = f_2(z)f_2(y)f_2(z)^{-1}f_2(y)^{-1} = g_2(z)k_zg_2(y)k_yk_z^{-1}$ $g_2(z)^{-1}k_y^{-1}g_2(y)^{-1} = g_2(z)g_2(y)g_2(z)^{-1}g_2(y)^{-1} = g_2(zyz^{-1}y^{-1})$.

Analogously, f_1 and g_1 coincide in the generators of Y_1. ∎

Lemma 7 *Let $\beta = (\beta_1, \beta_2) : (X_1, X_2, \delta) \twoheadrightarrow (M, P, \mu)$ be a central extension of a perfect precrossed module (M, P, μ). Then, the commutator precrossed submodule $[(X_1, X_2, \delta), (X_1, X_2, \delta)]$ is perfect, and $\beta : [(X_1, X_2, \delta), (X_1, X_2, \delta)] \twoheadrightarrow (M, P, \mu)$ is a central extension of (M, P, μ).*

Proof The second assertion is clear since $\beta([(X_1,X_2,\delta),(X_1,X_2,\delta)]) = [\beta(X_1,X_2,\delta),\beta(X_1,X_2,\delta)] = (M,P,\mu)$.

To see that $[(X_1,X_2,\delta),(X_1,X_2,\delta)]$ is perfect, we will prove the three following equalities:

$$\begin{aligned}
[X_1,X_1] &= [[X_1,X_1]\,[X_2,X_1]\,,[X_1,X_1]\,[X_2,X_1]] \\
[X_2,X_1] &= [[X_2,X_2]\,,[X_1,X_1]\,[X_2,X_1]] \\
[X_2,X_2] &= [[X_2,X_2]\,,[X_2,X_2]]
\end{aligned}$$

The restriction of β to the precrossed submodule $[(X_1,X_2,\delta),(X_1,X_2,\delta)]$ is a surjection, so for each element $x \in X_1$ there are elements $\widetilde{x} \in [X_1,X_1][X_2,X_1]$ and $a_x \in Ker\beta_1 \subset Z(X_1) \cap X_1^{X_2}$ such that $x = \widetilde{x}a_x$. For each element $u \in X_2$ there are also elements $\widetilde{u} \in [X_2,X_2]$ and $b_u \in Ker\beta_2 \subset Z(X_2) \cap st_{X_2}(X_1)$ such that $u = \widetilde{u}b_u$.

Now, for every $x,y \in X_1$ and $u,v \in X_2$ it is verified that $xyx^{-1}y^{-1} = \widetilde{x}\widetilde{y}\widetilde{x}^{-1}\widetilde{y}^{-1}, {}^{u}xx^{-1} = {}^{\widetilde{u}}\,\widetilde{x}\widetilde{x}^{-1}$ and $uvu^{-1}v^{-1} = \widetilde{u}\widetilde{v}\widetilde{u}^{-1}\widetilde{v}^{-1}$. ∎

Theorem 8 *Let $(V,R,\tau) \rightarrowtail (Y,F,\tau) \overset{\pi}{\twoheadrightarrow} (M,P,\mu)$ be a projective presentation of a perfect precrossed module (M,P,μ). Then the induced central extension*

$$[(Y,F,\tau),(Y,F,\tau)]\,/\,[(Y,F,\tau),(V,R,\tau)] \twoheadrightarrow (M,P,\mu)$$

with kernel $H_2(M,P,\mu)$ is the universal central extension of (M,P,μ).

Proof We will check the universal property for this induced central extension. Given a central extension $(X_1,X_2,\delta) \overset{\alpha}{\twoheadrightarrow} (M,P,\mu)$, there exists a morphism $\phi : (Y,F,\tau) \longrightarrow (X_1,X_2,\delta)$ such that $\pi = \alpha\phi$, since (Y,F,τ) is a projective precrossed module.

$$\begin{array}{ccc}
(Y,F,\tau) & & \\
\phi\downarrow & \searrow & \\
(X_1,X_2,\delta) & \overset{\alpha}{\twoheadrightarrow} & (M,P,\mu)
\end{array}$$

Since α is a central extension $\phi([(Y,F,\tau),(V,R,\tau)]) = 1$, so ϕ induces a morphism $\phi' : (Y,F,\tau)\diagup[(Y,F,\tau),(V,R,\tau)] \longrightarrow (X_1,X_2,\delta)$ that we restrict to $[(Y,F,\tau),(Y,F,\tau)]\diagup[(Y,F,\tau),(V,R,\tau)]$

$$\begin{array}{ccc}
[(Y,F,\tau),(Y,F,\tau)]\diagup[(Y,F,\tau),(V,R,\tau)] & & \\
\phi'\downarrow & & \searrow \\
(X_1,X_2,\delta) & \overset{\alpha}{\twoheadrightarrow} & (M,P,\mu)
\end{array}$$

Uniqueness follows from Lemma 6 and Lemma 7. ∎

If $\phi : (U_1, U_2, \omega) \twoheadrightarrow (M, P, \mu)$ is the universal central extension of a precrossed module (M, P, μ), then (M, P, μ) must be a perfect precrossed module; actually, (U_1, U_2, ω) must be a perfect precrossed module. If (U_1, U_2, ω) was not a perfect precrossed module then the canonical projection $(U_1, U_2, \omega) \stackrel{\pi}{\twoheadrightarrow} (U_1, U_2, \omega)_{ab}$ would be a non zero morphism. ϕ would not be universal since there exist at least two different morphisms into the second projection $(U_1, U_2, \omega)_{ab} \times (M, P, \mu) \twoheadrightarrow (M, P, \mu)$

$$\begin{array}{ccc} (U_1, U_2, \omega) & & \stackrel{\phi}{\twoheadrightarrow} (M, P, \mu) \\ \{\phi, 0\} \downarrow & \downarrow \{\phi, \pi\} & \| \\ (U_1, U_2, \omega)_{ab} \times (M, P, \mu) & & \twoheadrightarrow (M, P, \mu) \end{array}$$

Our next objective is to give an expression of the universal central extension of a perfect precrossed module in terms of non-abelian tensor products of groups.

Recall that given two groups M and N equipped with an action of M on N and an action of N on M the *tensor product* $M \otimes N$ is the group generated by the symbols $m \otimes n$, for $m \in M$ and $n \in N$, with relations

$$\begin{aligned} mm' \otimes n &= ({}^m m' \otimes {}^m n)(m \otimes n) \\ m \otimes nn' &= (m \otimes n)({}^n m \otimes {}^n n') \end{aligned}$$

for all $m, m' \in M$ and $n, n' \in N$, understanding that each group acts on itself by conjugation [5].

Given these actions, the free product $M * N$ acts on both M and N. The following proposition is a summary of the main properties of the tensor product in the case that the actions verify the compatibility conditions, that is, ${}^{({}^m n)}m' = {}^{mnm^{-1}}m'$ and ${}^{({}^n m)}n' = {}^{nmn^{-1}}n'$ if $m, m' \in M$ and $n, n' \in N$.

Proposition 9 *([5])*

(a) The free product $M * N$ acts on $M \otimes N$ by ${}^p(m \otimes n) = {}^p m \otimes {}^p n$, for $m \in M$, $n \in N$ and $p \in M * N$.

(b) There are homomorphisms $\lambda : M \otimes N \to M$ and $\lambda' : M \otimes N \to N$, $\lambda(m \otimes n) = m^n m^{-1}$, $\lambda'(m \otimes n) = {}^m n n^{-1}$ for $m \in M$ and $n \in N$.

(c) λ and λ' are crossed modules.

(d) If $l \in M \otimes N$, $m \in M$ and $n \in N$ then $\lambda(l) \otimes n = l\ {}^{n}l^{-1}$ and $m \otimes \lambda'(l) = {}^{m}l\ l^{-1}$.

(e) The actions of M on $Ker(\lambda')$ and N on $Ker(\lambda)$ are trivial.

(f) If $l, l' \in M \otimes N$, then $[l, l'] = \lambda(l) \otimes \lambda'(l')$.

If (M, P, μ) is a perfect precrossed module then P is a perfect group, and there exists the universal central extension of the group P [5]:

$$\zeta : P \otimes P \twoheadrightarrow P$$
$$p \otimes q \rightsquigarrow [p, q]$$

On the other hand, M is a normal subgroup of the semidirect product $M \rtimes P$. So we can consider the tensor product $M \otimes (M \rtimes P)$ with both of the groups acting on each other by conjugation.

Denote by $\nu : M \rtimes P \longrightarrow P$ the group homomorphism defined by $\nu(m, p) = \mu(m)p$, for $m \in M$ and $p \in P$. Plainly $(M \rtimes P, P, \nu)$ is a precrossed module and Example 1 is a special case of that. Furthermore it is a crossed module if and only if M is abelian and P operates trivially on it.

Proposition 10 *If (M, P, μ) is a precrossed module then $(M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu)$ is also a precrossed module, and $(\lambda, \zeta) : (M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu) \longrightarrow (M, P, \mu)$ is a precrossed module morphism.*

Proof There is an action of P on $M \otimes (M \rtimes P)$ given by ${}^{p}(m \otimes (n, q)) = {}^{p}m \otimes ({}^{p}n, pqp^{-1})$ for $p \in P$ and $m \otimes (n, q) \in M \otimes (M \rtimes P)$ that induces the one of $P \otimes P$ on $M \otimes (M \rtimes P)$ via ζ.

The homomorphism

$$\mu \otimes \nu : M \otimes (M \rtimes P) \longrightarrow P \otimes P$$

is defined by $(\mu \otimes \nu)(m \otimes (n, q)) = \mu(m) \otimes \mu(n)q$. We see that $(M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu)$ is a precrossed module by checking that $\mu \otimes \nu$ composed with the crossed module ζ is a precrossed module and $\mu \otimes \nu$ is P-equivariant.

$\zeta \circ (\mu \otimes \nu)({}^{p}(m \otimes (n, q))) = \zeta \circ (\mu \otimes \nu)({}^{p}m \otimes ({}^{p}n, pqp^{-1})) = [\mu({}^{p}m), \mu({}^{p}n)pqp^{-1}] = [p\mu(m)p^{-1}, p\mu(n)qp^{-1}] = p\,[\mu(m), \mu(n)q]\,p^{-1} = {}^{p}(\zeta \circ (\mu \otimes \nu)(m \otimes (n, q)))$ for $p \in P$ and $m \otimes (n, q) \in M \otimes (M \rtimes P)$.

Using Proposition 9 (b) we get a homomorphism $\lambda : M \otimes (M \rtimes P) \longrightarrow M$, given by $\lambda(m \otimes (n, q)) = mn\ {}^{q}m^{-1}n^{-1}$.

The pair (λ, ζ) is a precrossed module morphism. First we check that $\zeta \circ (\mu \otimes \nu) = \mu \circ \lambda$: $\zeta \circ (\mu \otimes \nu)\,(m \otimes (n,q)) = [\mu(m), \mu(n)q] = \mu\,(mn\ {}^{q}m^{-1}n^{-1}) = = (\mu \circ \lambda)\,(m \otimes (n,q))$ for $m \otimes (n,q) \in M \otimes (M \rtimes P)$.

On the other hand, $\lambda\,({}^{x\otimes y}(m \otimes (n,q))) = \lambda\left({}^{[x,y]}m \otimes ({}^{[x,y]}n, [x,y]\,q\,[x,y]^{-1})\right) = {}^{[x,y]}m^{[x,y]}n^{[x,y]q}m^{-1}\ {}^{[x,y]}n^{-1} = ^{[x,y]}(mn\ {}^{q}m^{-1}n^{-1}) = ^{\zeta(x\otimes y)}\lambda\,(m \otimes (n,q))$ for $x \otimes y \in P \otimes P$ and $m \otimes (n,q) \in M \otimes (M \rtimes P)$. ∎

Theorem 11 *If (M, P, μ) is a perfect precrossed module, then (λ, ζ) is the universal central extension of (M, P, μ).*

Proof (λ, ζ) is surjective since the precrossed module (M, P, μ) coincides with its commutator submodule.

λ and ζ are crossed modules by Proposition 9 (c), so $Ker(\lambda) \subset Z\,(M \otimes (M \rtimes P))$ and $Ker\,(\zeta) \subset Z\,(P \otimes P)$. From Proposition 9 (e) P acts trivially on $Ker(\lambda)$ and then $Ker(\lambda) \subset Inv(M \otimes (M \rtimes P))$, and $Ker(\zeta) \subset st_{P\otimes P}\,(M \otimes (M \rtimes P))$ since $P \otimes P$ acts on $M \otimes (M \rtimes P)$ via ζ.

The central extension (λ, ζ) is universal. Given an arbitrary central extension of (M, P, μ)

$$\psi = (\psi_1, \psi_2) : (X_1, X_2, \delta) \twoheadrightarrow (M, P, \mu),$$

let $s_1 : M \to X_1$ and $s_2 : P \to X_2$ be set theoretic sections of ψ_1 and ψ_2, respectively. For $m \in M$ and $(n,q) \in M \rtimes P$, the element $s_1(m)s_1(n)\ {}^{s_2(q)}s_1(m)^{-1}s_1(n)^{-1} \in X_1$ does not depend on the choice of s_1 and s_2: if s_1' and s_2' are other sections of ψ_1 and ψ_2, there are $x_m, x_n \in Ker(\psi_1) \subset Z(X_1) \cap Inv(X_1)$ such that $s_1'(m) = s_1(m)x_m$ and $s_1'(n) = s_1(n)x_n$; there is also an $y_q \in Ker(\psi_2) \subset Z(X_2) \cap st_{X_2}(X_1)$ such that $s_2'(q) = s_2(q)y_q$, so $s_1'(m)s_1'(n)^{s_2'(q)}s_1'(m)^{-1}s_1'(n)^{-1} = s_1(m)x_m s_1(n)x_n\ {}^{s_2(q)y_q}(x_m^{-1}s_1(m)^{-1})x_n^{-1}s_1(n)^{-1}$ $= s_1(m)s_1(n)^{s_2(q)}s_1(m)^{-1}s_1(n)^{-1}$.

It is clear that for $p, q \in P$ the element $[s_2(p), s_2(q)] \in X_2$ does not depend on the choice of s_2. It is straightforward that the maps $M \times (M \rtimes P) \to X_1, (m,(n,q)) \rightsquigarrow s_1(m)s_1(n)^{s_2(q)}s_1(m)^{-1}s_1(n)^{-1}$ and $P \times P \to X_2, (p,q) \rightsquigarrow [s_2(p), s_2(q)]$ factor throught $M \otimes (M \rtimes P)$ and $P \otimes P$ inducing two group homomorphisms

$$M \otimes (M \rtimes P) \xrightarrow{f_1} X_1$$
$$m \otimes (n,q) \rightsquigarrow s_1(m)s_1(n)^{s_2(q)}s_1(m)^{-1}s_1(n)^{-1}$$

$$P \otimes P \xrightarrow{f_2} X_2$$
$$p \otimes q \rightsquigarrow [s_2(p), s_2(q)]$$

It is verified that the pair $(f_1, f_2) : (M\otimes(M\rtimes P), P\otimes P, \mu\otimes\nu) \longrightarrow (X_1, X_2, \delta)$ is a precrossed module morphism, that makes the following triangle commute

$$\begin{array}{ccc} (M \otimes (M\rtimes P), P \otimes P, \mu \otimes \nu) & \longrightarrow & (M, P, \mu) \\ (f_1, f_2) \ \downarrow & & \nearrow \\ (X_1, X_2, \delta) & & \end{array}$$

It only remains to prove the uniqueness of (f_1, f_2). If $(f_1', f_2') : (M\otimes(M\rtimes P), P\otimes P, \mu \otimes \nu) \longrightarrow (X_1, X_2, \delta)$ is a precrossed module morphism such that $(\lambda, \zeta) = (\psi_1, \psi_2) \circ (f_1', f_2')$ then $f_2 = f_2'$, since ζ is the universal central extension of the group P. Since each generator $m \otimes (n, q) \in M \otimes (M\rtimes P)$ can be discomposed in $m\otimes(n, q) = m\otimes((1, q)({}^{q^{-1}}n, 1)) = (m\otimes(1, q))({}^{q}m\otimes(n, 1))$ we can restrict us to see that f_1 and f_1' coincide on the generators $m \otimes (n, 1)$ and $m \otimes (1, q)$ of $M \otimes (M\rtimes P)$.

Let

$$\lambda'' : M \otimes (M\rtimes P) \longrightarrow M\rtimes P$$

$$m \otimes (n, q) \rightsquigarrow (mn\ {}^{q}m^{-1}n^{-1}, 1)$$

be the composite of λ in Proposition 9 (b) with the canonical inclusion $M \hookrightarrow M \rtimes P$. Since $(n, 1) \in Im\,(\lambda'')$ and $m \in Im\,(\lambda)$ there exist elements $l, l' \in M \otimes (M\rtimes P)$ such that $\lambda''\,(l') = (n, 1)$ and $\lambda\,(l) = m$. These elements determine $a, b \in Ker\,(\psi_1) \subset Z(X_1) \cap Inv(X_1)$ verifying $f_1'(l) = f_1(l)a$ and $f_1'(l') = f_1(l')b$. It follows from Proposition 9 (f) that $f_1'\,(m \otimes (n, 1)) = f_1'\,(\lambda(l) \otimes \lambda''\,(l')) = f_1'\,([l, l']) = [f_1'\,(l)\,, f_1'\,(l')] = [f_1(l)a, f_1(l')b] = [f_1\,(l)\,, f_1\,(l')] = f_1\,([l, l']) = f_1\,(\lambda(l) \otimes \lambda''\,(l')) = f_1\,(m \otimes (n, 1))$.

On the other hand, using Proposition 9 (d) and the fact that the action of $M\rtimes P$ on $M \otimes (M\rtimes P)$ extends the action of P on $M \otimes (M\rtimes P)$, we get $f_1'\,(m \otimes (1, q)) = f_1'\,(\lambda(l) \otimes (1, q)) = f_1'(l\ {}^{(1,q)}l^{-1}) = f_1'(l\ {}^{q}l^{-1}) = f_1'(l)\ {}^{f_2(q)}f_1'(l^{-1}) = f_1(l)a\ {}^{f_2(q)}(a^{-1}f_1(l^{-1})) = f_1(l)\ {}^{f_2(q)}f_1(l^{-1}) = f_1(l\ {}^{q}l^{-1}) = f_1(l\ {}^{(1,q)}l^{-1}) = f_1\,(\lambda(l) \otimes \lambda''\,(l')) = f_1\,(m \otimes (1, q))$. ∎

Example 12 *Let $\zeta : P\otimes P \twoheadrightarrow P$ be the universal central extension of a perfect group P. If we consider the group P as a perfect precrossed module we get the universal central extensions*

$$(1, H_2(P), i) \rightarrowtail (1, P \otimes P, i) \overset{(1,\zeta)}{\twoheadrightarrow} (1, P, i)$$

$$(H_2(P), 1, 1) \rightarrowtail (P \otimes P, 1, 1) \overset{(\zeta,1)}{\twoheadrightarrow} (P, 1, 1)$$

which generalize the universal central extension in the category of groups.

Now we state a sufficient condition for a central extension to be universal:

Lemma 13 *Let $\beta = (\beta_1, \beta_2) : (X_1, X_2, \delta) \twoheadrightarrow (M, P, \mu)$ be a central extension of a precrossed module (M, P, μ). If (X_1, X_2, δ) is a perfect precrossed module and every central extension of (X_1, X_2, δ) splits, then β is the universal central extension of (M, P, μ).*

Proof For each central extension of (M, P, μ)

$$\alpha = (\alpha_1, \alpha_2) : (Y_1, Y_2, \sigma) \twoheadrightarrow (M, P, \mu)$$

construct the pullback of α and β:

$$\begin{array}{ccc} (X_1 \underset{M}{\times} Y_1, X_2 \underset{P}{\times} Y_2, \delta \times \sigma) & \xrightarrow{\pi_1} & (X_1, X_2, \delta) \\ \pi_2 \downarrow & & \downarrow \beta \\ (Y_1, Y_2, \sigma) & \xrightarrow{\alpha} & (M, P, \mu) \end{array}$$

It is easy to prove that π_1 is a central extension using the fact that α is also central, then π_1 is split by hypothesis. So we get a morphism of central extensions from β to α. It is unique since (X_1, X_2, δ) is perfect. ∎

4 Application to Algebraic K-Theory

For a perfect crossed module (T, G, ∂), the universal central extension of (T, G, ∂) in the category of crossed modules (see [15] and [17]) is

$$H_2^{CCG}(T, G, \partial) \rightarrowtail (T \otimes G, G \otimes G, \partial \otimes id) \twoheadrightarrow (T, G, \partial)$$

where $H_2^{CCG}(T, G, \partial)$ denotes the second homological invariant of the crossed module (T, G, ∂) defined in [6]. Since the perfect crossed module (T, G, ∂) is also a perfect precrossed module, it has an universal (precrossed) central extension

$$H_2(T, G, \partial) \rightarrowtail (T \otimes (T \rtimes G), G \otimes G, \partial \otimes \nu) \twoheadrightarrow (T, G, \partial)$$

It is clear that the quotient by the Peiffer elements of the universal precrossed central extension of (T, G, ∂) is universal for central extensions of crossed modules of (T, G, ∂), so we get the following result:

Theorem 14 *Let (T, G, ∂) be a perfect crossed module. Then*

$$(T \otimes G, G \otimes G, \partial \otimes id) \cong (T \otimes (T \rtimes G) / \langle T \otimes (T \rtimes G), T \otimes (T \rtimes G) \rangle, G \otimes G, \partial \otimes \nu)$$

$$H_2^{CCG}(T, G, \partial) \cong H_2(T, G, \partial) / (\langle T \otimes (T \rtimes G), T \otimes (T \rtimes G) \rangle, 1, 1)$$

Algebraic K-theory will provide us an useful example of a perfect crossed module whose universal precrossed central extension is different from its universal crossed central extension.

It is known that for a ring R, the K-theory group $K_2(R)$ has the following algebraic interpretation [14]: let $St(R)$ be the Steinberg group generated by $x_{ij}(r)$, with i, j a pair of distinct integers and $r \in R$, subject to the relations $x_{ij}(r)x_{ij}(s) = x_{ij}(r+s)$, $[x_{ij}(r), x_{kl}(s)] = 1$ if $j \neq k$ and $i \neq l$, $[x_{ij}(r), x_{jk}(s)] = x_{ik}(rs)$ if $i \neq k$. If we denote by $\varphi_R : St(R) \to GL(R)$ the natural group homomorphism from the Steinberg group to the general linear group, then $K_2(R) = Ker(\varphi_R)$. The image of φ_R is the perfect group $E(R)$, the subgroup of $GL(R)$ generated by the elementary matrices. Since $\varphi_R : St(R) \twoheadrightarrow E(R)$ is the universal central extension of $E(R)$, it is deduced that $K_2(R)$ is isomorphic to the second homology group of $E(R)$.

Now, given a two-sided ideal I of R, and Φ a functor on the category of rings with values in the category of groups, the relative group $\Phi(I)$ can be defined as follows [14,18]: denote by D the pullback of the natural ring homomorphism $R \twoheadrightarrow R/I$.

$$\begin{array}{ccc} D & \xrightarrow{p_1} & R \\ p_2 \downarrow & & \downarrow \\ R & \longrightarrow & R/I \end{array}$$

The projections p_1 and p_2 are split by the diagonal homomorphism $\Delta : R \to D$. These ring homomorphisms induce the group homomorphisms

$$\Phi(D) \underset{p_{2*}}{\overset{p_{1*}}{\rightrightarrows}} \Phi(R) \quad (\text{split by } \Delta_*)$$

$\Phi(I)$ is defined as the kernel of p_{1*}. So $E(I) = Ker(p_{1*|E(D)})$, $St(I) = Ker(p_{1*|St(D)})$ and $K_2(I) = Ker(p_{1*|K_2(D)})$.

The following theorem is a generalization in $\mathcal{PCM}$ of the classical algebraic interpretation of $K_2(R)$ due a Kervaire [12]:

Theorem 15 *For each two-sided ideal I of a ring R, there exists a perfect crossed module $(E(I), E(R), i)$, whose universal precrossed central extension* *is*

$$(K_2(I), K_2(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i)$$

Proof $E(I)$ is equal to the normal subgroup $E(R) \cap GL(I)$ of $E(R)$. Since $E(I) = [E(R), E(I)]$, $(E(I), E(R), i)$ is a perfect crossed module, where i denotes the inclusion of $E(I)$ in $E(R)$. To construct its universal central extension, note that there exists a commutative diagram with exact rows and columns

$$\begin{array}{ccccc}
K_2(I) & \rightarrowtail & K_2(D) & \overset{\curvearrowleft}{\rightrightarrows} & K_2(R) \\
\downarrow & & \downarrow & & \downarrow \\
St(I) & \rightarrowtail & St(D) & \overset{\curvearrowleft}{\rightrightarrows} & St(R) \\
\downarrow & & \downarrow & & \downarrow \\
GL(I) & \rightarrowtail & GL(D) & \overset{\curvearrowleft}{\rightrightarrows} & GL(R)
\end{array}$$

where the first row homomorphisms are induced by those from the second and third rows. Exactness in the second and third rows implies that $K_2(I)$ is the kernel of the induced homomorphism $St(I) \to GL(D)$.

Taking the Moore complex of this diagram, which is an exact sequence of truncated simplicial groups, we get the precrossed module extension

$$(K_2(I), K_2(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i)$$

To prove that it is a central extension, note that $K_2(D)$ is the center of $St(D)$, so $K_2(I) \rtimes K_2(R) \cong K_2(D) = Z(St(D)) \cong Z(St(I) \rtimes St(R))$, from where is easily deduced that $(K_2(I), K_2(R), \gamma) \subset Z(St(I), St(R), \gamma)$.

This central extension is universal: $St(D)$ is a superperfect group, that is, it is perfect and has trivial second homology group. So $St(I) \rtimes St(R) \cong St(D) = [St(D), St(D)] \cong [St(I) \rtimes St(R), St(I) \rtimes St(R)] = [St(I), St(I)][St(R), St(I)] \rtimes [St(R), St(R)]$ and $(St(I), St(R), \gamma)$ is a perfect precrossed module. From the description of the second homology of a precrossed module given in [1, Theorem 4.3.], $H_2(St(I), St(R), \gamma) \cong (\Sigma, H_2(St(R)), \sigma_*) = 0$, where $\Sigma = Ker(H_2(St(D)) \longrightarrow H_2(St(R)))$. In analogy with group theory we will say that $(St(I), St(R), \gamma)$ is a *superperfect precrossed module.* Its universal central extension is, of course, the identity

$$(St(I), St(R), \gamma) \xrightarrow{id} (St(I), St(R), \gamma)$$

The theorem now follows from Lemma 13 applied to the extension

$$(St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i) \quad \blacksquare$$

Corollary 16 *In the conditions of the previous theorem we have the isomorphisms*

$$H_2(E(I), E(R), i) \cong (K_2(I), K_2(R), \gamma)$$

$$(St(I), St(R), \gamma) \cong (E(I) \otimes E(D), E(R) \otimes E(R), i \otimes \nu)$$

Remark 17 *(i) The perfect precrossed module $(St(I), St(R), \gamma)$ is not always a crossed module. Loday [13] and Keune [11] introduce a relative Steinberg group that is denoted by $St(R, I)$, verifying $St(R, I) \cong St(I)/C(I)$, where $C(I)$ is the Peiffer subgroup of $(St(I), St(R), \gamma)$. Swan shows in [19] that the Peiffer subgroup $C(I)$ can be non trivial. If $f : \mathbb{Z}[t] \to \mathbb{Z}$ where $f(t) = 0$, and we write $R = \mathbb{Z}[t]$ and $I = Kerf$, then $1 \neq [x_{12}(0, t), x_{21}(0, t)] \in C(I)$.*

(ii) Gilbert says in [9] that the universal crossed central extension of $(E(I), E(R), i)$ is $(St(R, I), St(R), \overline{\gamma})$. He makes use of the isomorphism $E(I) \otimes E(R) \cong St(R, I)$ taken from [8]. However, Theorem 15 together with the previous observation shows that the kernel of this central extension is not the Gilbert's second homology of $(E(I), E(R), i)$, because it coincides with $H_2(E(I), E(R), i)$ [1, Theorem 4.2.], which is the kernel of the universal precrossed central extension of $(E(I), E(R), i)$. The correct kernel of $(St(R, I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i)$ is the second homology of crossed modules of [6] or [16],

$$H_2^{CCG}(E(I), E(R), i) \cong H_2^{GL}(E(I), E(R), i) \cong (K_2(R, I), K_2(R), \overline{\gamma})$$

where $K_2(R, I)$ denotes the second relative K-theory group introduced by Loday [13] and Keune [11].

Moreover, he claims that the second homology of crossed modules he defines, coincides with that defined in [16] for aspherical crossed modules. The example above shows that this affirmation is wrong.

(iii) Anyway, the universal precrossed central extension and the universal crossed central extension of a perfect crossed module may coincide. If we take a perfect group P, and consider the perfect crossed module (P, P, id), by an easy calculation fulfilled in [9], we get $H_2(P, P, id) \cong (H_2(P) \oplus (H_1(P) \otimes H_1(P)), H_2(P), \sigma_*) = (H_2(P), H_2(P), id) = H_2^{CCG}(P, P, id)$, so we have a morphism of extensions

$$\begin{array}{ccccc} H_2(P,P,id) & \rightarrowtail & (P \otimes (P \rtimes P), P \otimes P, id \otimes \nu) & \twoheadrightarrow & (P,P,id) \\ \downarrow & & \downarrow & & \downarrow id \\ H_2^{CCG}(P,P,id) & \rightarrowtail & (P \otimes P, P \otimes P, id) & \twoheadrightarrow & (P,P,id) \end{array}$$

inducing an isomorphism in the kernels, so both of the extensions are isomorphic.

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