On universal central extensions of precrossed and crossed modules

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Abstract

We study the connection between universal central extensions in the categories of precrossed and crossed modules. They are compared with several kinds of universal central extensions in the categories of groups, epimorphisms of groups, groups with operators and modules over a group. We study the relationship between the homologies defined in these categories. Applications to relative algebraic K-theory are also obtained.

Key words: universal central extension, precrossed module, crossed module, perfect group, relative central extension, central equivariant extension.
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1 Introduction

Universal central extensions in the category of precrossed modules $\mathcal{PCM}$ were first studied in [4]. However, universal central extensions in the subcategory of crossed modules $\mathcal{CM} \subset \mathcal{PCM}$ have been studied widely before [7,11,19–21].

In [4] it was shown that for a perfect crossed module (T, G, ∂) it makes sense to consider the universal central extensions of (T, G, ∂) in $\mathcal{PCM}$ and in $\mathcal{CM}$, since in general these two extensions do not coincide. We will denote them by $U(T, G, \partial)$ and $U_\mathcal{C}(T, G, \partial)$ respectively. An interesting example from relative algebraic K-theory can be used to illustrate this situation (see Example 11 for details): for a two-sided ideal I of a ring R there is a perfect crossed module $(E(I), E(R), i)$ of elementary matrices, whose universal central extension in $\mathcal{PCM}$ is

$$(K_2(I), K_2(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i)$$

but its universal central extension in $\mathcal{CM}$ is

$$(K_2(R, I), K_2(R), \overline{\gamma}) \rightarrowtail (St(R, I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i)$$

where $St(I)$ and $K_2(I)$ are the Stein relativizations of $St(R)$ and $K_2(R)$, while $St(R, I)$ and $K_2(R, I)$ denote the relative groups of Loday and Keune.

In this paper we begin by going deep in this situation. The central result in Section 3 shows that for a perfect crossed module (T, G, ∂), there is an epimorphism $U(T, G, \partial) \twoheadrightarrow U_\mathcal{C}(T, G, \partial)$ between the universal central extensions of (T, G, ∂), which is simultaneously a universal central extension in $\mathcal{PCM}$ and a Peiffer quotient. For example, there is a universal central extension $(St(I), St(R), \gamma) \twoheadrightarrow (St(R, I), St(R), \overline{\gamma})$ which is also a Peiffer quotient. As a consequence we deduce some expressions for the second homology in $\mathcal{PCM}$ and in $\mathcal{CM}$ of (T, G, ∂), and a sufficient and necessary condition for (T, G, ∂) in order to have the same universal central extension in $\mathcal{PCM}$ and $\mathcal{CM}$.

Universal central extensions in $\mathcal{PCM}$ and $\mathcal{CM}$ clearly generalize the classical universal central extension of a perfect group [14]. In fact, both of them contain as second component this universal central extension. In Section 4 we will deal with the interpretation of the first components of the universal central extensions in $\mathcal{PCM}$ and $\mathcal{CM}$. We compare them with the universal relative central extensions of Loday [16], the universal central equivariant extensions of Cegarra and Inassaridze [8] and the universal G-central extensions of Dennis and Igusa [9].

The relationship between the homology theories of all these categories is established. As an application to algebraic K-theory we give some examples of universal extensions involving the above mentioned K-theory groups and the Steinberg additive group and K-theory groups defined in [9,13].

2 Preliminaries on Precrossed Modules

In this section we recall some standard results on precrossed modules that will be used in the sequel.

A *precrossed module* (M, P, μ) is a group homomorphism $\mu : M \to P$ together with an action of P on M, denoted $^p m$ for $m \in M$ and $p \in P$, such that $\mu(^p m) = p\mu(m)p^{-1}$ for all $m \in M$ and $p \in P$. (M, P, μ) is called a *crossed module* if in addition it satisfies the Peiffer identity $^{\mu(m)} m' = mm'm^{-1}$ for every $m, m' \in M$.

We denote by $\mathcal{PCM}$ the category of precrossed modules, and by $\mathcal{CM}$ the category of crossed modules.

Precrossed modules naturally arise in several algebraic and topological contexts. These are some of the standard examples of precrossed modules.

Example 1 *i) A pair of groups (G, N), that is, a group G and a normal subgroup N, can be considered as a crossed module of the form (N, G, i), where G acts on N by conjugation and $i : N \hookrightarrow G$ denotes the inclusion.*

ii) Groups with operators can be considered as precrossed modules: if Γ is a group and G is a Γ-group, then $(G, \Gamma, 1)$ is a precrossed module, which is a crossed module when G is a Γ-module.

A precrossed module morphism $(\psi_1, \psi_2) : (Y, X, \delta) \to (M, P, \mu)$ is said to be *injective (surjective)* if ψ_1 and ψ_2 are injective (surjective) homomorphisms. A surjective precrossed module morphism $(Y, X, \delta) \twoheadrightarrow (M, P, \mu)$ is also called an *extension of* (M, P, μ).

A *precrossed submodule* (N, Q, μ') of a precrossed module (M, P, μ) is a precrossed module such that N and Q are, respectively, subgroups of M and P, the action of Q on N is induced by the one of P on M and $\mu' = \mu_{|N}$. It is said to be a *normal precrossed submodule* if besides N and Q are normal in M and P, $^p n \in N$ and $^q mm^{-1} \in N$ for all $p \in P$, $q \in Q$, $m \in M$ and $n \in N$.

We can define for a normal precrossed submodule (N, Q, μ) of (M, P, μ), the *quotient precrossed module* $(M, P, \mu)/(N, Q, \mu)$ as $(M/N, P/Q, \overline{\mu})$, where the homomorphism $\overline{\mu}$ is induced by μ and P/Q acts on M/N by $^{pQ} mN = (^p m)N$ for $p \in P$ and $m \in M$.

We call the *Peiffer subgroup* of a precrossed module (M, P, μ) the subgroup $\langle M, M \rangle$ of M generated by the Peiffer elements $m_1 m_2 m_1^{-1\,\mu(m_1)} m_2^{-1}$, with $m_1, m_2 \in M$. It is a normal subgroup of M, and the precrossed submodule $(\langle M, M \rangle, 1, 1)$ is the smallest one making the quotient $(M, P, \mu)/(\langle M, M \rangle,$

$1, 1)$ a crossed module. So we have defined a functor $(-)_{Peiff} : \mathcal{PCM} \to \mathcal{CM}$, which assigns to a precrossed module (M, P, μ) the quotient $(M, P, \mu)_{Peiff} = (M, P, \mu)/(\langle M, M \rangle, 1, 1)$. $(-)_{Peiff}$ is left adjoint to the inclusion functor $i : \mathcal{CM} \to \mathcal{PCM}$. It is usually called the *Peiffer quotient* or *Peiffer abelianisation functor* though a crossed module is not abelian in general.

The *kernel* $\mathrm{Ker}(\psi)$ of a precrossed module morphism $\psi = (\psi_1, \psi_2) : (Y, X, \delta) \to (M, P, \mu)$ is the normal precrossed submodule $(\mathrm{Ker}(\psi_1), \mathrm{Ker}(\psi_2), \delta)$ of (Y, X, δ).

The *center* $\mathrm{Z}(M, P, \mu)$ of a precrossed module (M, P, μ) is the normal precrossed submodule $(\mathrm{Inv}(M) \cap \mathrm{Z}(M), \mathrm{St}_P(M) \cap \mathrm{Z}(P), \mu)$, where $\mathrm{Z}(M), \mathrm{Z}(P)$ denote the centers of M and P, $\mathrm{St}_P(M) = \{p \in P \mid^p m = m \text{ for all } m \in M\}$ and $\mathrm{Inv}(M) = \{m \in M \mid \mu(m) \in \mathrm{St}_P(M) \text{ and } ^p m = m \text{ for all } p \in P\}$.

An *abelian* precrossed module is a precrossed module (M, P, μ) such that $(M, P, \mu) = \mathrm{Z}(M, P, \mu)$. Equivalently M and P are abelian groups and P acts trivially on M.

If (N, Q, μ) and (R, K, μ) are normal precrossed submodules of (M, P, μ), we define the *commutator precrossed submodule* $[(N, Q, \mu), (R, K, \mu)]$ as the normal precrossed submodule $([Q, R] [K, N] [N, R], [Q, K], \mu)$ of (M, P, μ), where $[Q, R]$ is the normal subgroup of M generated by the elements $^q r r^{-1}$ with $q \in Q, r \in R$, $[K, N]$ denotes the normal subgroup of M generated by the elements $^k n n^{-1}$ with $k \in K, n \in N$ and $[N, R]$ and $[Q, K]$ denote the usual commutator subgroups of N with R and Q with K.

In particular, the *commutator precrossed submodule* of a precrossed module (M, P, μ) is $[(M, P, \mu), (M, P, \mu)] = ([M, M] [P, M], [P, P], \mu)$. It is the smallest normal precrossed submodule of (M, P, μ) making the quotient an abelian precrossed module.

The inclusion of abelian precrossed modules $\mathcal{APCM}$ in $\mathcal{PCM}$ has as left adjoint the *abelianisation functor* $ab : \mathcal{PCM} \to \mathcal{APCM}$ which assigns to a precrossed module (M, P, μ) the abelian precrossed module $(M, P, \mu)_{ab} = (M/([M, M] [P, M]), P/ [P, P], \overline{\mu})$.

In [3] it was shown that $\mathcal{PCM}$ is an algebraic category with enough projective objects, and, for a precrossed module (M, P, μ) and an integer $n \geq 1$, the *n-th homology precrossed module* with trivial coefficients $H_n(M, P, \mu)$ was introduced.

The first homology functor is simply the abelianisation functor $H_1(M, P, \mu) = (M, P, \mu)_{ab}$. Remark that this property of H_1 is, mutatis mutandis, the same for each homology theory used in this paper. In [4] it was shown how to compute the second homology precrossed module $H_2(M, P, \mu)$ by means of a Hopf type formula in terms of a projective presentation of (M, P, μ).

3 Universal central extensions of a perfect crossed module

A precrossed module extension $(\psi_1, \psi_2) : (Y, X, \delta) \twoheadrightarrow (M, P, \mu)$ is called *central* if $\mathrm{Ker}(\psi_1, \psi_2) \subset \mathrm{Z}(Y, X, \delta)$.

The central extension (ψ_1, ψ_2) is said to be *universal* if for every central extension $(\phi_1, \phi_2) : (S, H, \tau) \twoheadrightarrow (M, P, \mu)$ of (M, P, μ) there exists a unique morphism $(\varphi_1, \varphi_2) : (Y, X, \delta) \to (S, H, \tau)$ rendering commutative the following diagram

$$
\begin{array}{ccc}
(Y, X, \delta) & \xrightarrow{(\psi_1, \psi_2)} & (M, P, \mu) \,. \\
{\scriptstyle (\varphi_1, \varphi_2)}\big\downarrow & {\scriptstyle (\phi_1, \phi_2)}\nearrow & \\
(S, H, \tau) & &
\end{array}
\tag{1}
$$

If there is a universal central extension of (M, P, μ), then it is unique up to isomorphism. It will be denoted by $U(M, P, \mu) \twoheadrightarrow (M, P, \mu)$.

There only exist universal central extensions of perfect precrossed modules. A precrossed module (M, P, μ) is called *perfect* if $H_1(M, P, \mu) = 0$, and *superperfect* if, furthermore, $H_2(M, P, \mu) = 0$. Remark that (M, P, μ) is perfect if and only if it is equal to its commutator precrossed module.

We recall from [4] and [2] the following characterization of universal central extensions:

Theorem 2 *i) A precrossed module (M, P, μ) admits a universal central extension if and only if, (M, P, μ) is a perfect precrossed module.*

ii) A central extension $(Y, X, \delta) \twoheadrightarrow (M, P, \mu)$ is universal if and only if, (Y, X, δ) is a perfect precrossed module and every central extension of (Y, X, δ) splits.

iii) The kernel of the universal central extension of a perfect precrossed module (M, P, μ) is isomorphic to $H_2(M, P, \mu)$.

We deduce a homological characterization of universal central extensions:

Corollary 3 *A central extension $(Y, X, \delta) \twoheadrightarrow (M, P, \mu)$ is universal if and only if, (Y, X, δ) is a superperfect precrossed module.*

Proof. This follows from Theorem 2 ii): if (Y, X, δ) is a superperfect precrossed module, it is isomorphic to $U(Y, X, \delta)$. Hence by Diagram (1) every central extension of (Y, X, δ) splits.

Conversely, if every central extension of (Y, X, δ) splits, then the identity $\mathrm{Id} : (Y, X, \delta) \to (Y, X, \delta)$ is the universal central extension of (Y, X, δ): for a central extension $(\alpha_1, \alpha_2) : (L, C, \omega) \twoheadrightarrow (Y, X, \delta)$ there is a splitting which commutes the following triangle

$$
\begin{array}{ccc}
(Y, X, \delta) & \xlongequal{\ \mathrm{Id}\ } & (Y, X, \delta) \, . \\
\big\downarrow & \nearrow_{(\alpha_1, \alpha_2)} & \\
(L, C, \omega) & &
\end{array}
$$

By [4, Lemma 6] the splitting is unique, and by Theorem 2 iii) $H_2(Y, X, \delta) = \mathrm{Ker}(\mathrm{Id}) = 0$. $\square$

As was told in the introduction and proved in [4] a perfect crossed module (T, G, ∂) has different universal central extensions in the categories $\mathcal{PCM}$ and $\mathcal{CM}$ (a construction of the last one can be found in [19]). We will denote by $U(T, G, \partial) \twoheadrightarrow (T, G, \partial)$ the first one and by $U_{\mathcal{C}}(T, G, \partial) \twoheadrightarrow (T, G, \partial)$ the second one.

Lemma 4 $U_{\mathcal{C}}(T, G, \partial) = (U(T, G, \partial))_{Peiff}$.

Proof. $U(T, G, \partial)$ is an initial object in the category of central extensions of (T, G, ∂) in $\mathcal{PCM}$. The Peiffer quotient functor is left adjoint to the inclusion functor $\mathcal{CM} \to \mathcal{PCM}$ and these two functors restrict to adjoint functors between the categories of central extensions of (T, G, ∂) in $\mathcal{PCM}$ and $\mathcal{CM}$. Hence $(U(T, G, \partial))_{Peiff}$ is an initial object in the category of central extensions of (T, G, ∂) in $\mathcal{CM}$. $\square$

Theorem 5 *The universal central extension of $U_{\mathcal{C}}(T, G, \partial)$ in $\mathcal{PCM}$ is*

$$
H_2(U_{\mathcal{C}}(T, G, \partial)) \rightarrowtail U(T, G, \partial) \twoheadrightarrow U_{\mathcal{C}}(T, G, \partial).
$$

Proof. $U(T, G, \partial) \twoheadrightarrow U_{\mathcal{C}}(T, G, \partial)$ is a central extension since its kernel is contained in the kernel of $U(T, G, \partial) \twoheadrightarrow (T, G, \partial)$. The proof follows from Corollary 3, since $U(T, G, \partial)$ is a superperfect precrossed module. $\square$

We will denote by $H_2^{CCG}(T, G, \partial)$ the second homological invariant of a crossed module defined in [7] by Carrasco, Cegarra and R.-Grandjeán. $H_2^{CCG}(T, G, \partial)$ is the kernel of $U_{\mathcal{C}}(T, G, \partial) \twoheadrightarrow (T, G, \partial)$ [21]. We can summarise the results

above in the following diagram with exact rows and columns:

$$
\begin{array}{ccc}
H_2(U_{\mathcal{C}}(T,G,\partial)) & \!=\!= & H_2(U_{\mathcal{C}}(T,G,\partial)) \\
\downarrow & & \downarrow \\
H_2(T,G,\partial) \rightarrowtail & U(T,G,\partial) \twoheadrightarrow & (T,G,\partial) \\
\downarrow & & \downarrow & \| \\
H_2^{CCG}(T,G,\partial) \rightarrowtail & U_{\mathcal{C}}(T,G,\partial) \twoheadrightarrow & (T,G,\partial)
\end{array}
\qquad (2)
$$

where the central row and column are universal central extensions in $\mathcal{PCM}$ and the bottom row is universal in $\mathcal{CM}$.

We deduce an expression for the second precrossed module homology of a perfect crossed module:

Corollary 6 *For a perfect crossed module (T,G,∂), the second homology $H_2(T,G,\partial)$ is the pullback of the morphisms $U(T,G,\partial) \twoheadrightarrow U_{\mathcal{C}}(T,G,\partial)$ and $H_2^{CCG}(T,G,\partial) \rightarrowtail U_{\mathcal{C}}(T,G,\partial)$, that is*

$$
H_2(T,G,\partial) = H_2^{CCG}(T,G,\partial) \underset{U_{\mathcal{C}}(T,G,\partial)}{\times} U(T,G,\partial).
$$

Both universal central extensions in $\mathcal{PCM}$ and $\mathcal{CM}$ have expressions in terms of the non-abelian tensor product of groups defined by Brown and Loday [6]. We recall them in the following lemmas.

Lemma 7 *[19] Let (T,G,∂) be a perfect crossed module. Then*

$$
U_{\mathcal{C}}(T,G,\partial) \cong (T \otimes G, G \otimes G, \partial \otimes \mathrm{Id}).
$$

Lemma 8 *[4] Let (M,P,μ) be a perfect precrossed module. Denote by $\nu : M \rtimes P \to P$ the group homomorphism defined by $\nu(m,p) = \mu(m)p$ for $m \in M$ and $p \in P$. Then*

$$
U(M,P,\mu) \cong (M \otimes (M \rtimes P), P \otimes P, \mu \otimes \nu).
$$

Remark 9 *Combining Theorem 5 and Lemmas 7 and 8, for each perfect crossed module (T,G,∂) we get the universal central extension in $\mathcal{PCM}$*

$$
H_2(T{\otimes}G, G{\otimes}G, \partial{\otimes}\mathrm{Id}) \rightarrowtail (T{\otimes}(T{\rtimes}G), G{\otimes}G, \partial{\otimes}\nu) \twoheadrightarrow (T{\otimes}G, G{\otimes}G, \partial{\otimes}\mathrm{Id}),
$$

and a short exact sequence

$$
H_2(T \otimes G, G \otimes G, \partial \otimes \mathrm{Id}) \rightarrowtail H_2(T,G,\partial) \twoheadrightarrow H_2^{CCG}(T,G,\partial),
$$

where $H_2(T \otimes G, G \otimes G, \partial \otimes \mathrm{Id}) = (\langle T \otimes (T \rtimes G), T \otimes (T \rtimes G)\rangle, 1, 1)$.

7

Corollary 10 *For a perfect crossed module* (T, G, ∂) *there are isomorphisms of groups*

$$T \otimes (T \rtimes G) \cong (T \otimes G) \otimes ((T \otimes G) \rtimes (G \otimes G))$$

$$G \otimes G \cong (G \otimes G) \otimes (G \otimes G).$$

Proof. By Theorem 5, $U(T \otimes G, G \otimes G, \partial \otimes \mathrm{Id}) \cong (T \otimes (T \rtimes G), G \otimes G, \partial \otimes \nu)$.

On the other hand Lemma 8 yields $U(T \otimes G, G \otimes G, \partial \otimes \mathrm{Id}) \cong ((T \otimes G) \otimes ((T \otimes G) \rtimes (G \otimes G)), (G \otimes G) \otimes (G \otimes G), (\partial \otimes \mathrm{Id}) \otimes \tilde{\nu})$, where $\tilde{\nu} : (T \otimes G) \rtimes (G \otimes G) \to G \otimes G$ denotes the group homomorphism defined by $\tilde{\nu}((t \otimes g_1), (g_2 \otimes g_3)) = (\partial(t) \otimes g_1)(g_2 \otimes g_3)$ for $t \in T$ and $g_1, g_2, g_3 \in G$. $\square$

Example 11 *[4] For each two-sided ideal I of a ring R there exists a perfect crossed module $(E(I), E(R), i)$, where $E(R)$ is the subgroup of $GL(R)$ generated by the elementary matrices, and $E(I)$ is the Stein relativization ([18,22]) of the functor $E(-)$. Recall that given a functor Φ on the category of rings with values in the category of groups, if we denote by D the kernel pair of the natural ring homomorphism $R \twoheadrightarrow R/I$*

$$\begin{array}{ccc} D & \xrightarrow{\ p_1\ } & R \\ {\scriptstyle p_2}\downarrow & & \downarrow \\ R & \longrightarrow & R/I \end{array}$$

the Stein relative group $\Phi(I)$ is defined as the kernel of the induced homomorphism of groups $p_{1*} : \Phi(D) \to \Phi(R)$.

The universal central extension in $\mathcal{PCM}$ of $(E(I), E(R), i)$ is

$$(K_2(I), K_2(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (E(I), E(R), i)$$

where $St(R)$ and $K_2(R)$ denote the Steinberg group and the second K-theory group of the ring R, and $St(I)$ and $K_2(I)$ denote their Stein relativizations [4]. $K_2(I)$ is the relative K-theory group introduced by Milnor in [18]. Recall that the Steinberg group *is the group generated by the elements $x_{ij}(r)$, with i, j a pair of distinct integers and $r \in R$, subject to the relations $x_{ij}(r)x_{ij}(s) = x_{ij}(r + s)$, $[x_{ij}(r), x_{kl}(s)] = 1$ if $j \neq k$ and $i \neq l$, $[x_{ij}(r), x_{jk}(s)] = x_{ik}(rs)$ if $i \neq k$, where $r, s \in R$.*

On the other hand, the universal central extension of $(E(I), E(R), i)$ in $\mathcal{CM}$ is (see [11])

$$(K_2(R, I), K_2(R), \overline{\gamma}) \rightarrowtail (St(R, I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i)$$

where $St(R, I)$ denotes the relative Steinberg group defined by Keune [15] and $K_2(R, I)$ denotes the second relative K-theory group introduced by Loday [16] and Keune [15]. From Lemma 4 and Diagram (2) we deduce that $(St(R, I), St(R), \overline{\gamma})$ is the Peiffer quotient of $(St(I), St(R), \gamma)$, with kernel $H_2(St(R, I), St(R), \gamma) \cong (\langle St(I), St(I) \rangle, 1, 1)$.

Theorem 5 says that $U(St(R, I), St(R), \overline{\gamma}) = (St(I), St(R), \gamma)$, that is, the following central extension is universal in $\mathcal{PCM}$

$$H_2(St(R, I), St(R), \gamma) \rightarrowtail (St(I), St(R), \gamma) \twoheadrightarrow (St(R, I), St(R), \overline{\gamma}).$$

Furthermore, there is an extension of abelian precrossed modules

$$H_2(St(R, I), St(R), \gamma) \rightarrowtail (K_2(I), K_2(R), \gamma) \twoheadrightarrow (K_2(R, I), K_2(R), \gamma)$$

so the abelian group $K_2(R, I)$ is isomorphic to the quotient $\frac{K_2(I)}{\langle St(I), St(I) \rangle}$.

From Corollary 6, $K_2(I)$ is the pullback of the group homomorphisms $St(I) \twoheadrightarrow St(R, I)$ and $K_2(R, I) \rightarrowtail St(R, I)$, and then

$$K_2(I) = K_2(R, I) \underset{St(R, I)}{\times} St(I).$$

By Lemmas 7 and 8 we get the following expressions for the Steinberg groups involved in these extensions

$$St(R) \cong E(R) \otimes E(R)$$

$$St(R, I) \cong E(I) \otimes E(R) \ [10]$$

$$St(I) \cong E(I) \otimes (E(I) \rtimes E(R)) \cong St(R, I) \otimes (St(R, I) \rtimes St(R)).$$

In Remark 33 we will give another example of a perfect crossed module with different universal central extension in $\mathcal{CM}$ and $\mathcal{PCM}$.

Let $(V, R, \tau) \rightarrowtail (W, F, \tau) \twoheadrightarrow (T, G, \partial)$ be a projective presentation in $\mathcal{PCM}$ of a perfect crossed module (T, G, ∂). It was shown in [4] that the universal central extension of (T, G, ∂) in $\mathcal{PCM}$ can be expressed as

$$\left(\frac{[W, W][F, W]}{[W, V][F, V][R, W]}, \frac{[F, F]}{[F, R]}, \overline{\tau} \right) \twoheadrightarrow (T, G, \partial).$$

On the other hand, since the inclusion of categories $i : \mathcal{CM} \to \mathcal{PCM}$ preserves surjections, the Peiffer quotient $(-)_{Peiff} : \mathcal{PCM} \to \mathcal{CM}$, maps projective precrossed modules to projective crossed modules, and then we have a projective presentation $\frac{(V, R, \tau)}{(\langle W, W \rangle, 1, 1)} \rightarrowtail \frac{(W, F, \tau)}{(\langle W, W \rangle, 1, 1)} \twoheadrightarrow (T, G, \partial)$ of (T, G, ∂) in $\mathcal{CM}$ ($\langle W, W \rangle \subset V$ since $(W, F, \tau)/(V, R, \tau)$ is a crossed module). If we denote $\widehat{W} = \frac{W}{\langle W, W \rangle}$ and $\widehat{V} = \frac{V}{\langle W, W \rangle}$, the projective presentation is $(\widehat{V}, R, \tau) \rightarrowtail$

$(\widehat{W}, F, \tau) \twoheadrightarrow (T, G, \partial)$ and following [7], the universal central extension in $\mathcal{CM}$ of (T, G, ∂) is

$$\left(\frac{[F, \widehat{W}]}{[F, \widehat{V}][R, \widehat{W}]}, \frac{[F, F]}{[F, R]}, \overline{\tau} \right) \twoheadrightarrow (T, G, \partial).$$

Next we give an expression for the kernel $H_2(U_\mathcal{C}(T, G, \partial))$ of both the universal central extension $U(T, G, \partial) \twoheadrightarrow U_\mathcal{C}(T, G, \partial)$ and the morphism $H_2(T, G, \partial) \twoheadrightarrow H_2^{CCG}(T, G, \partial)$.

Proposition 12 *Let $(V, R, \tau) \rightarrowtail (W, F, \tau) \twoheadrightarrow (T, G, \partial)$ be a projective presentation of a perfect crossed module (T, G, ∂) in $\mathcal{PCM}$.*

Then

$$H_2(U_\mathcal{C}(T, G, \partial)) \cong \left(\frac{\langle W, W \rangle}{[W, V][F, V][R, W] \cap \langle W, W \rangle}, 1, 1 \right).$$

Proof. $(\langle W, W \rangle, 1, 1) \subset [(W, F, \tau), (W, F, \tau)]$, since $\frac{(W, F, \tau)}{[(W, F, \tau), (W, F, \tau)]}$ is a crossed module. Then $\langle W, W \rangle \subset [W, W][F, W]$, and there is a commutative diagram with exact rows

$$([W, V][F, V][R, W]) \cap \langle W, W \rangle \rightarrowtail [W, V][F, V][R, W] \longrightarrow [F, \widehat{V}][R, \widehat{W}]$$

$$\langle W, W \rangle \rightarrowtail [W, W][F, W] \xrightarrow{\ p\ } [F, \widehat{W}]$$

where $p : W \twoheadrightarrow \widehat{W}$ denotes the Peiffer quotient.

The kernel of $\frac{[W, W][F, W]}{[W, V][F, V][R, W]} \twoheadrightarrow \frac{[F, \widehat{W}]}{[F, \widehat{V}][R, \widehat{W}]}$ is $\frac{\langle W, W \rangle}{([W, V][F, V][R, W]) \cap \langle W, W \rangle}$, and the proposition follows from Theorem 5. $\square$

Remark 13 *Using Hopf formulae for $H_2(T, G, \partial)$ [4] and $H_2^{CCG}(T, G, \partial)$ [7], the first component of $H_2(U_\mathcal{C}(T, G, \partial)) \rightarrowtail H_2(T, G, \partial) \twoheadrightarrow H_2^{CCG}(T, G, \partial)$ can be expressed in terms of the projective presentation $(V, R, \tau) \rightarrowtail (W, F, \tau) \twoheadrightarrow (T, G, \partial)$ as*

$$\frac{\langle W, W \rangle}{([W, V][F, V][R, W]) \cap \langle W, W \rangle} \rightarrowtail \frac{V \cap ([F, W][W, W])}{[F, V][R, W][W, V]} \longrightarrow \frac{\frac{V}{\langle W, W \rangle} \cap [F, \frac{W}{\langle W, W \rangle}]}{[F, \frac{V}{\langle W, W \rangle}][\frac{W}{\langle W, W \rangle}, R]}.$$

Next we state a sufficient and necessary condition under which a perfect crossed module has the same universal central extension in $\mathcal{PCM}$ and $\mathcal{CM}$:

Corollary 14 *With the notations of Proposition 12, $U_\mathcal{C}(T, G, \partial)$ is isomorphic to $U(T, G, \partial)$ if and only if, $\langle W, W \rangle \subset [W, V][F, V][R, W]$.*

Proof. From Remark 13, $\langle W, W \rangle \subset [W,V][F,V][R,W]$ if and only if, the morphism $H_2(T,G,\partial) \twoheadrightarrow H_2^{CCG}(T,G,\partial)$ is an isomorphism. The corollary follows from Diagram (2). $\square$

4 Connection with other universal extensions

Let (Q,N) denote a group epimorphism $\nu : N \twoheadrightarrow Q$. Recall that a *relative central extension* of (Q,N) by an abelian group L is an exact sequence [16]

$$0 \to L \to M \xrightarrow{\lambda} N \xrightarrow{\nu} Q \to 1$$

where (M,N,λ) is a crossed module inducing a trivial action on $L \cong \mathrm{Ker}(\lambda)$.

A *morphism* between two relative central extensions $0 \to L \to M \xrightarrow{\lambda} N \xrightarrow{\nu} Q \to 1$ and $0 \to L' \to M' \xrightarrow{\lambda'} N \xrightarrow{\nu} Q \to 1$ of (Q,N) is a crossed module morphism $(f, \mathrm{Id}) : (M,N,\lambda) \to (M',N,\lambda')$. Clearly such a morphism induces a group homomorphism between the kernels L and L' and a commutative diagram

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & L & \longrightarrow & M & \xrightarrow{\lambda} & N & \xrightarrow{\nu} & Q & \longrightarrow & 1 \\
& & \downarrow & & {\scriptstyle f}\downarrow & & \| & & \| & & \\
0 & \longrightarrow & L' & \longrightarrow & M' & \xrightarrow{\lambda'} & N & \xrightarrow{\nu} & Q & \longrightarrow & 1.
\end{array}
$$

A relative central extension $0 \to L \to M \xrightarrow{\lambda} N \xrightarrow{\nu} Q \to 1$ of (Q,N) is called *universal* if there exists one and only one morphism from it to another relative central extension of (Q,N).

If a group Γ acts on a group G (that is, G is a Γ-group) G is called Γ-*perfect* if $G = [\Gamma, G][G,G]$, where $[\Gamma, G]$ denotes the normal subgroup of G generated by the elements $^x g g^{-1}$ with $x \in \Gamma, g \in G$ and $[G,G]$ denotes the usual commutator subgroup of G.

There exists a universal relative central extension of (Q,N) if and only if $\mathrm{Ker}(\nu)$ is a N-perfect group [16, Théorème 2]. The kernel of this extension is the relative homology group $H_3(Q,N;\mathbb{Z})$.

4.1 Universal central extensions in $\mathcal{PCM}$ and universal relative central extensions of groups

From Lemma 8 the second component $P \otimes P \twoheadrightarrow P$ of the universal central extension of a perfect precrossed module (M,P,μ) is the universal central

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extension of the perfect group P [6]. In this subsection we will prove that the first component $M \otimes (M \rtimes P) \twoheadrightarrow M$ is related to a universal relative central extension.

Given a precrossed module central extension

$$(Y, X, \delta) \xrightarrow{(\psi_1, \psi_2)} (M, P, \mu)$$

we will construct a relative central extension of groups as follows: the extension of groups $Y \rtimes X \xrightarrow{\psi_1 \rtimes \psi_2} M \rtimes P$ is central and induces an action of $M \rtimes P$ on $Y \rtimes X$ by $^{(\psi_1(y_1), \psi_2(x_1))}(y, x) = \left(y_1 \; ^{x_1}y \; ^{x_1 x x_1^{-1}}y_1^{-1}, x_1 x x_1^{-1}\right)$ for $x, x_1 \in X$, $y, y_1 \in Y$. With this action Y is a $M \rtimes P$-subgroup with action $^{(\psi_1(y_1), \psi_2(x_1))}y = y_1 \; ^{x_1}y \; y_1^{-1}$. Denoting by $i : M \hookrightarrow M \rtimes P$ the inclusion $i(m) = (m, 1)$, we obtain a crossed module $(Y, M \rtimes P, i\psi_1)$, which induces a trivial action of P on $\mathrm{Ker}(\psi_1)$: if $x \in X$ and $k \in \mathrm{Ker}(\psi_1)$, then $^{\psi_2(x)}k = \; ^{(1, \psi_2(x))}k = \; ^x k = k$, since $\mathrm{Ker}(\psi_1) \subset Inv(Y)$. The exact sequence

$$0 \to \mathrm{Ker}(\psi_1) \to Y \xrightarrow{i\psi_1} M \rtimes P \xrightarrow{\pi} P \to 1$$

is a relative central extension of $(P, M \rtimes P)$, where $\pi : M \rtimes P \to P$ denotes the canonical projection.

On the other hand, given a relative central extension of $(P, M \rtimes P)$

$$0 \to \mathrm{Ker}(\lambda) \to C \xrightarrow{\lambda} M \rtimes P \xrightarrow{\pi} P \to 1$$

observe that $\mathrm{Im}(\lambda) \subset M$ and then it is easy to verify the existence of a precrossed module central extension

$$(C, P, \mu\lambda) \xrightarrow{(\lambda, \mathrm{Id})} (M, P, \mu)$$

where C is a P-group via $j : P \hookrightarrow M \rtimes P$.

Theorem 15 *If $(\psi_1, \psi_2) : (Y, X, \delta) \twoheadrightarrow (M, P, \mu)$ is the universal central extension of a perfect precrossed module (M, P, μ), then*

$$0 \to \mathrm{Ker}(\psi_1) \to Y \xrightarrow{i\psi_1} M \rtimes P \xrightarrow{\pi} P \to 1$$

is the universal relative central extension of $(P, M \rtimes P)$.

Proof. For a relative central extension $0 \to \mathrm{Ker}(\lambda) \to C \xrightarrow{\lambda} M \rtimes P \xrightarrow{\pi} P \to 1$ of $(P, M \rtimes P)$ there is a unique precrossed module morphism (α, ψ_2) such that

the following triangle commutes

$$(Y, X, \delta) \xrightarrow{(\psi_1, \psi_2)} (M, P, \mu)\,.$$

with (α, ψ_2) going down to $(C, P, \mu\lambda)$ and (λ, Id) the diagonal.

The group homomorphism α induces a morphism of relative central extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\psi_1) & \longrightarrow & Y & \xrightarrow{i\psi_1} & M \rtimes P & \xrightarrow{\pi} & P & \longrightarrow & 1 \\
& & \downarrow & & \alpha\downarrow & & \| & & \| & & \\
0 & \longrightarrow & \mathrm{Ker}(\lambda) & \longrightarrow & C & \xrightarrow{\lambda} & M \rtimes P & \xrightarrow{\pi} & P & \longrightarrow & 1
\end{array}
$$

since $i\psi_1 = \lambda\alpha$ and $\alpha\left(^{(\psi_1(y), \psi_2(x))}y_1\right) = \alpha\left(y\,^x y_1\,y^{-1}\right) = \alpha(y)\,^{\psi_2(x)}\alpha(y_1)\,\alpha(y)^{-1} = {}^{\lambda\alpha(y)}\left(^{(0,\psi_2(x))}\alpha(y_1)\right) = {}^{i\psi_1(y)\,(0,\psi_2(x))}\alpha(y_1) = {}^{(\psi_1(y), \psi_2(x))}\alpha(y_1)$ for $x \in X$ and $y, y_1 \in Y$.

Finally α is unique: if there were another morphism $\beta : Y \to C$ between these relative central extensions, then there would be a commutative triangle of precrossed module morphisms

$$(Y, X, \delta) \xrightarrow{(\psi_1, \psi_2)} (M, P, \mu)$$

with (β, ψ_2) going down to $(C, P, \mu\lambda)$ and (λ, Id) the diagonal.

since $\psi_2\delta = \mu\lambda\alpha = \mu\psi_1 = \mu\lambda\beta$, and $\beta\left(^x y\right) = \beta\left(^{(0,\psi_2(x))}y\right) = {}^{(0,\psi_2(x))}\beta(y) = {}^{\psi_2(x)}\beta(y)$ for $x \in X$ and $y \in Y$. By the universal property of (Y, X, δ) we deduce that $(\alpha, \psi_2) = (\beta, \psi_2)$, and so $\alpha = \beta$. $\square$

Remark 16 *The universal central extension (ψ_1, ψ_2) of a perfect precrossed module (M, P, μ)*

$$
\begin{array}{ccc}
M \otimes (M \rtimes P) & \xrightarrow{\psi_1} & M \\
\mu\otimes\nu\downarrow & & \downarrow\mu \\
P \otimes P & \xrightarrow{\psi_2} & P
\end{array}
$$

contains as first component the universal relative central extension $i\psi_1$ of $(P, M \rtimes P)$ and as second component the universal central extension ψ_2 of the group P.

Example 17 *We pointed out in Example 11 that $(St(I), St(R), \gamma)$ is the universal central extension in $\mathcal{PCM}$ of $(E(I), E(R), i)$ and $(St(R, I), St(R), \overline{\gamma})$. From Theorem 15 we obtain the following universal relative central extensions*

$$0 \to K_2(I) \to St(I) \to E(D) \to E(R) \to 1$$

13

$$0 \to \langle St(I), St(I) \rangle \to St(I) \to St(R, I) \rtimes St(R) \to St(R) \to 1$$

where D denotes the kernel pair of the ring homomorphism $R \twoheadrightarrow \frac{R}{I}$ *(see Example 11)*.

Corollary 18 *For a perfect precrossed module* (M, P, μ)

$$H_2(M, P, \mu) = (H_3(P, M \rtimes P; \mathbb{Z}), H_2(P), \mu_*).$$

Proof. This follows from Theorem 2 iii), Remark 16 and [16, Théorème 2]. $\square$

Remark 19 *Some groups of classes of central extensions in* $\mathcal{PCM}$ *and of relative central extensions of groups are isomorphic [1]. As a consequence, cohomology of precrossed modules and relative cohomology of groups are also related.*

4.2 Universal central extensions in $\mathcal{CM}$ and universal relative central extensions of groups

Given a relative central extension

$$0 \to L \to M \xrightarrow{\lambda} N \xrightarrow{\nu} Q \to 1$$

we can construct a central extension (λ, Id) in $\mathcal{CM}$

of $(\mathrm{Ker}(\nu), N, j)$, where j is the inclusion of the normal subgroup $\mathrm{Ker}(\nu) \lhd N$.

On the other hand, given a central extension in $\mathcal{CM}$

$$(V, U, \omega) \xrightarrow{(\varphi_1, \varphi_2)} (\mathrm{Ker}(\nu), N, j)$$

we construct a relative central extension of (Q, N)

$$0 \to \mathrm{Ker}(\varphi_1) \to V \xrightarrow{\varphi_2 \omega} N \xrightarrow{\nu} Q \to 1$$

where the action of N on V is induced by the one of (V, U, ω) since $\mathrm{Ker}(\varphi_2)$ acts trivially on V, and $\varphi_2 \omega$ is easily seen to be a crossed module.

In the following theorem we show that universal relative central extensions of group epimorphisms (Q, N) with N a perfect group are contained as first

14

components of certain universal central extensions in $\mathcal{CM}$. Remark that N perfect implies Q perfect.

Theorem 20 *Let $\nu : N \twoheadrightarrow Q$ be a group epimorphism such that N is a perfect group and $\mathrm{Ker}(\nu)$ is N-perfect. If $(\varphi_1, \varphi_2) : (V, U, \omega) \twoheadrightarrow (\mathrm{Ker}(\nu), N, j)$ is the universal central extension in $\mathcal{CM}$ of the perfect crossed module $(\mathrm{Ker}(\nu), N, j)$, then*

$$0 \to \mathrm{Ker}(\varphi_1) \to V \xrightarrow{j\varphi_1} N \xrightarrow{\nu} Q \to 1$$

is the universal relative central extension of (Q, N).

Proof. $(\mathrm{Ker}(\nu), N, j)$ is clearly a perfect crossed module. For a relative central extension

$$0 \to L \to M \xrightarrow{\lambda} N \xrightarrow{\nu} Q \to 1$$

there is a unique crossed module morphism (α, φ_2)

$$
(V, U, \omega) \xrightarrow{(\varphi_1, \varphi_2)} (\mathrm{Ker}(\nu), N, j) \, .
$$
$$(\alpha, \varphi_2) \downarrow \qquad \nearrow (\lambda, \mathrm{Id})$$
$$(M, N, \lambda)$$

The group homomorphism α induces a morphism of relative central extensions

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & \mathrm{Ker}(\varphi_1) & \longrightarrow & V & \xrightarrow{j\varphi_1} & N & \xrightarrow{\nu} & Q & \longrightarrow & 1 \\
& & \downarrow & & \alpha\downarrow & & \| & & \| & & \\
0 & \longrightarrow & L & \longrightarrow & M & \xrightarrow{\lambda} & N & \xrightarrow{\nu} & Q & \longrightarrow & 1
\end{array}
$$

which is unique: if there were another morphism $\beta : V \to M$ between these relative central extensions then there would be a commutative triangle of crossed module morphisms

$$
(V, U, \omega) \xrightarrow{(\varphi_1, \varphi_2)} (\mathrm{Ker}(\nu), N, j) \, .
$$
$$(\beta, \varphi_2) \downarrow \qquad \nearrow (\lambda, \mathrm{Id})$$
$$(M, N, \lambda)$$

By uniqueness of (α, φ_2) we deduce that $\alpha = \beta$. $\square$

Corollary 21 *For a group epimorphism $\nu : N \twoheadrightarrow Q$ with N perfect and $\mathrm{Ker}(\nu)$ N-perfect, the universal relative central extension of (Q, N) is*

$$0 \to H_3(Q, N; \mathbb{Z}) \to \mathrm{Ker}(\nu) \otimes N \to N \xrightarrow{\nu} Q \to 1.$$

Proof. This follows from Lemma 7, Theorem 20 and [16, Théorème 2]. $\square$

Remark 22 *i) Given a perfect precrossed module (M, P, μ) we have a perfect crossed module $(M, M \rtimes P, i)$ whose universal central extension in $\mathcal{CM}$ has as first component $M \otimes (M \rtimes P) \twoheadrightarrow M$, which is the same as the first component of the universal central extension in $\mathcal{PCM}$ of (M, P, μ). Note that applying Theorem 15 or Theorem 20 we get that $M \otimes (M \rtimes P) \twoheadrightarrow M$ is the universal relative central extension of the epimorphism $(P, M \rtimes P)$.*

ii) The universal central extension in $\mathcal{CM}$ of a perfect crossed module (N, G, i), with i injective

$$
\begin{array}{ccc}
N \otimes G & \xrightarrow{\varphi_1} & N \\
\scriptstyle{i \otimes \mathrm{Id}} \downarrow & & \downarrow \scriptstyle{i} \\
G \otimes G & \xrightarrow{\varphi_2} & G
\end{array}
$$

contains as first component the universal relative central extension of the projection $G \twoheadrightarrow \frac{G}{N}$ and as second component the universal central extension of the group G.

Corollary 23 *Let $N \lhd G$ be a normal subgroup of a group G such that (N, G, i) is perfect. The second homology in $\mathcal{CM}$ of (N, G, i) is*

$$
H_2^{CCG}(N, G, i) = \left(H_3\left(\frac{G}{N}, G; \mathbb{Z}\right), H_2(G), i_* \right).
$$

Proof. This follows from Theorem 2 iii), Remark 22 and [16, Théorème 2]. $\square$

Example 24 *The universal central extension in $\mathcal{CM}$ of the perfect crossed module $(E(I), E(R), i)$ is (see Example 11)*

$$
(K_2(R, I), K_2(R), \overline{\gamma}) \rightarrowtail (St(R, I), St(R), \overline{\gamma}) \twoheadrightarrow (E(I), E(R), i).
$$

Applying Remark 22 we obtain the universal relative central extension of the group epimorphism $E(R) \twoheadrightarrow \frac{E(R)}{E(I)}$

$$
0 \to K_2(R, I) \to St(R, I) \to E(R) \to \frac{E(R)}{E(I)} \to 1.
$$

4.3 Universal central extensions in $\mathcal{PCM}$ and universal central equivariant extensions of Γ-groups

Recall from [8] that a *central equivariant extension of a Γ-group G* is a short exact sequence of Γ-groups

$$
0 \to A \to E \xrightarrow{p} G \to 1
$$

such that $A \subset \mathrm{Z}(E)$ and Γ acts trivially on A.

A central equivariant extension $0 \to A \to E \xrightarrow{p} G \to 1$ of G is called *universal* if for each equivariant extension $0 \to A' \to E' \xrightarrow{p'} G \to 1$ of G there is a unique Γ-homomorphism $\varphi : E \to E'$ such that the following diagram commutes

$$
\begin{array}{ccccccccc}
0 & \longrightarrow & A & \longrightarrow & E & \xrightarrow{\ p\ } & G & \longrightarrow & 1 \\
  &   &   &   & \varphi \downarrow &   & \| &   & \\
0 & \longrightarrow & A' & \longrightarrow & E' & \xrightarrow{\ p'\ } & G & \longrightarrow & 1.
\end{array}
$$

The universal central equivariant extension of a Γ-group G exists if and only if, G is a Γ-perfect group, and the kernel of this universal central extension is the equivariant homology group $H_2^{\Gamma}(G)$ [8, Theorem 7.5].

Given a central extension in $\mathcal{PCM}$

$$
(Y, X, \delta) \xrightarrow{(\psi_1, \psi_2)} (M, P, \mu)
$$

we have a central equivariant extension of the P-group M

$$
0 \to \mathrm{Ker}(\psi_1) \to Y \xrightarrow{\psi_1} M \to 1
$$

since the central homomorphism ψ_2 induces an action of P on Y such that ψ_1 is a P-homomorphism.

On the other hand, given a precrossed module (M, P, μ), a central equivariant extension $0 \to A \to E \xrightarrow{p} M \to 1$ of the P-group M gives rise to a central extension (p, Id) of (M, P, μ) in $\mathcal{PCM}$

$$
\begin{array}{ccccc}
A & \rightarrowtail & E & \xrightarrow{\ p\ } & M \\
\downarrow &   & \downarrow{\scriptstyle \mu p} &   & \downarrow{\scriptstyle \mu} \\
1 & \longrightarrow & P & \xupdownequals{\ \mathrm{Id}\ } & P \ .
\end{array}
$$

We will use these constructions to prove the following

Theorem 25 *If $(\psi_1, \psi_2) : (Y, X, \delta) \twoheadrightarrow (M, P, \mu)$ is the universal central extension in $\mathcal{PCM}$ of a perfect precrossed module (M, P, μ), then*

$$
0 \to \mathrm{Ker}(\psi_1) \to Y \xrightarrow{\psi_1} M \to 1
$$

is the universal central equivariant extension of the P-group M.

Proof. For a central equivariant extension $0 \to A \to E \xrightarrow{p} M \to 1$ of the P-group M there is a unique precrossed module morphism (α, ψ_2) such that

the diagram

$$(Y, X, \delta) \xrightarrow{(\psi_1, \psi_2)} (M, P, \mu)$$

$$(\alpha, \psi_2) \Big\downarrow \qquad \nearrow (p, \mathrm{Id})$$

$$(E, P, \mu p)$$

commutes. Hence α is a P-homomorphism and it is the only one such that the following triangle commutes

$$Y \xrightarrow{\psi_1} M$$
$$\alpha \Big\downarrow \quad \nearrow p$$
$$E$$

by the uniqueness of (α, ψ_2). $\quad \square$

Example 26 *Since $(St(I), St(R), \gamma)$ is the universal central extension in $\mathcal{PCM}$ of both $(E(I), E(R), i)$ and $(St(R, I), St(R), \overline{\gamma})$, from Theorem 25 we obtain the universal central equivariant extension of the $E(R)$-group $E(I)$*

$$0 \to K_2(I) \to St(I) \to E(I) \to 1$$

and the universal central equivariant extension of the $St(R)$-group $St(R, I)$

$$0 \to \langle St(I), St(I) \rangle \to St(I) \to St(R, I) \to 1.$$

Remark 27 *The universal central extension in $\mathcal{PCM}$*

$$M \otimes (M \rtimes P) \xrightarrow{\psi_1} M$$
$$\mu \otimes \nu \Big\downarrow \qquad \qquad \Big\downarrow \mu$$
$$P \otimes P \xrightarrow{\psi_2} P$$

of a perfect precrossed module (M, P, μ), contains as first component the universal central equivariant extension of the P-group M (which is the same as the universal relative central extension of the epimorphism $(P, M \rtimes P)$) and as second component the universal central extension of the group P.

Corollary 28 *For a perfect precrossed module (M, P, μ)*

$$H_2(M, P, \mu) = (H_2^P(M), H_2(P), \mu_*).$$

Proof. This follows from Theorem 2 iii), Remark 27 and [8, Theorem 7.5]. $\quad \square$

Example 29 *Following Example 26 we conclude that*

$$H_2^{E(R)}(E(I)) = K_2(I)$$

$$H_2^{St(R)}(St(R, I)) = \langle St(I), St(I) \rangle$$
$$H_2^{St(R)}(St(I)) = 0.$$

Remark 30 *On the other hand, if Γ is a perfect group and G a Γ-perfect group, the universal central equivariant extension of G is the first component of the universal central extension in $\mathcal{PCM}$ of the perfect precrossed module $(G, \Gamma, 1)$.*

Corollary 31 *Let Γ be a perfect group and G a Γ-perfect group. The universal central equivariant extension of G is*

$$H_3(\Gamma, G \rtimes \Gamma; \mathbb{Z}) \rightarrowtail G \otimes (G \rtimes \Gamma) \longrightarrow G.$$

Proof. This follows from Lemma 8, Remark 30 and Corollary 18. $\square$

4.4 Universal central extensions in $\mathcal{CM}$ and universal central module extensions

The notions of universal G-central extension and G-perfect module were first developed in [9,13] with the aim of computing some K-theory and homology groups. Later they were applied in [5,17] to Homotopy theory.

The *G-center* of a G-module A is the submodule of elements on which G acts trivially, and therefore a *G-central extension* of A is a short exact sequence of G-modules $K \rightarrowtail E \twoheadrightarrow A$ such that K is a trivial G-module.

In [9, Theorem 1.5] Dennis and Igusa prove that a G-module A has a universal G-central extension if and only if A is G-perfect (that is, a perfect G-group).

Given a group G and a G-module A we can consider the crossed module $(A, G, 0)$. This crossed module is perfect if and only if G is a perfect group and A is G-perfect. The following occurs to the universal central extension in $\mathcal{CM}$ of $(A, G, 0)$:

Theorem 32 *Let $(\varphi_1, \varphi_2) : (V, U, \omega) \twoheadrightarrow (A, G, 0)$ be the universal central extension in $\mathcal{CM}$ of the perfect crossed module $(A, G, 0)$. Then*

$$0 \to \operatorname{Ker}(\varphi_1) \to V \xrightarrow{\varphi_1} A \to 0$$

is the universal G-central extension of A.

Proof. This is analogous to the proof of Theorem 25. Remark that the Peiffer

identity for (V, U, ω) implies that V is abelian, since $\omega = 0 \otimes \mathrm{Id} = 0$ by Lemma 7. $\square$

Remark 33 $H_2^{CCG}(A, G, 0) = (H_1(G; A), H_2(G), 0)$ *[21]. So the universal central extension in $\mathcal{CM}$ of the perfect crossed module $(A, G, 0)$ is*

$$(H_1(G; A), H_2(G), 0) \rightarrowtail (A \otimes G, G \otimes G, 0) \twoheadrightarrow (A, G, 0)$$

while its universal central extension in $\mathcal{PCM}$ is

$$(H_2^G(A), H_2(G), 0) \rightarrowtail (A \otimes (A \rtimes G), G \otimes G, 1) \twoheadrightarrow (A, G, 0).$$

Therefore the Peiffer quotient of the universal central equivariant extension of the G-group A

$$H_2^G(A) \rightarrowtail A \otimes (A \rtimes G) \twoheadrightarrow A$$

is the universal G-central extension of A

$$H_1(G; A) \rightarrowtail A \otimes G \twoheadrightarrow A.$$

Using Guin's isomorphism [12] $A \otimes G \cong A \otimes_G IG$ we can identify this sequence with the one obtained in [5].

Specializing Diagram (2) to this situation and combining Remark 9 with Corollary 28 we obtain the universal central equivariant extension of the $(G \otimes G)$-group $A \otimes G$

$$H_2^{G \otimes G}(A \otimes G) \rightarrowtail A \otimes (A \rtimes G) \twoheadrightarrow A \otimes G$$

and a short exact sequence between the kernels of these universal extensions

$$H_2^{G \otimes G}(A \otimes G) \rightarrowtail H_2^G(A) \twoheadrightarrow H_1(G; A)$$

where $H_2^{G \otimes G}(A \otimes G) \cong \langle A \otimes (A \rtimes G), A \otimes (A \rtimes G) \rangle$.

Example 34 *Given a ring R and an R-bimodule A, let $M(A)$ denote the inductive limit of the groups $M_n(A)$ of matrices of order n with coefficients in A. The group of elementary matrices $E(R)$ acts by conjugation on $M(A)$.*

The Steinberg additive group $St(R, A)$ [9] is defined as the $St(R)$-module generated by the symbols $z_{ij}(a)$, with i, j a pair of distinct integers and $a \in A$, subject to the relations $z_{ij}(a) + z_{ij}(b) = z_{ij}(a+b)$, $x_{kh}(r)z_{ij}(a) = z_{ij}(a)$ if $i \neq h$ and $j \neq k$, $x_{ki}(r)z_{ij}(a) = z_{ij}(a) + z_{kj}(ra)$ if $j \neq k$, $x_{jh}(r)z_{ij}(a) = z_{ij}(a) + z_{ih}(-ar)$ if $i \neq h$, where $a, b \in A$, $r \in R$ and the elements $x_{ij}(r)$ are the generators of $St(R)$ (Example 11).

Consider the $E(R)$-module $M_0(A)$ generated by the matrices $E_{ij}(a)$, with $a \in A$ and $i \neq j$, where $E_{ij}(a) \in M(A)$ is the matrix with exactly one nonzero entry

a in the (i, j) position. $M_0(A)$ is $E(R)$-perfect (and therefore $St(R)$-perfect). Its universal $St(R)$-central extension is

$$K_2(R, A) \rightarrowtail St(R, A) \xrightarrow{\phi} M_0(A)$$

where $\phi(z_{ij}(a)) = E_{ij}(a)$ [9, Theorem 1.8], and $K_2(R, A)$ is a relative K-theory group introduced in [9] following ideas from A. Hatcher, isomorphic to the Hochschild homology $H_1(R, A)$ of R with coefficients in A.

We remark that $\phi : St(R, A) \twoheadrightarrow M_0(A)$ is also a universal $E(R)$-central extension [9, Part B].

Following Theorem 32 the universal central extension in $\mathcal{CM}$ of the perfect crossed module $(M_0(A), E(R), 0)$ is

$$(K_2(R, A), K_2(R), 0) \rightarrowtail (St(R, A), St(R), 0) \longrightarrow (M_0(A), E(R), 0)$$

and $H_2^{CCG}(M_0(A), E(R), 0) = (H_1(R, A), K_2(R), 0)$.

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