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The class of all natural implicative expansions of Kleene's strong logic functionally equivalent to Łukasiewicz's 3-valued logic Ł3

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Abstract We consider the logics determined by the set of all natural implicative expansions of Kleene's strong 3-valued matrix (with both only one and two designated values) and select the class of all logics functionally equivalent to Łukasiewicz's 3-valued logic Ł3. The concept of a "natural implicative matrix" is based upon the notion of a "natural conditional" defined in Tomova (2012), "A lattice of implicative extensions of regular Kleene's logics", *Reports on Mathematical Logic* 47, pp. 173-182.

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1 Introduction

Kleene's strong 3-valued matrix MK3 was defined in [22] in the context of the treatment of partial recursive functions. The matrix MK3 (our label) can be defined as shown in Definition 2.4 below. The connectives are conjunction,

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disjunction and negation. We can take either 2 as the only designated value or else both 1 and 2. In the former case, 1 can be interpreted as neither truth nor falsity; in the latter, as both truth and falsity. The value 2 is, of course, truth, while 0 is falsity.

There are several possibilities for expanding the matrix MK3 with a conditional connective. For example, we can define the conditional with disjunction and negation similarly as in classical logic. Then, if 2 is the only designated value, the set of valid formulas is empty, but if 1 and 2 are designated, all (and only all) tautologies of classical logic are validated. Alternatively, the conditional can be introduced by means of an independent function. In this way, for example, Łukasiewicz's 3-valued matrix MŁ3 or the 3-valued matrix MRM3 can be defined (MRM3 is the matrix determining the 3-valued extension of the quasi-relevant logic RM; cf. [1], [9]). MŁ3 (resp., MRM3) is defined upon MK3 with only one (resp., two) designated value.

In this paper, we shall consider all natural implicative expansions of MK3 with both one and two designated values. The notion of a natural conditional is here understood as an extension of that introduced in [41] and can be defined as shown in Definition 2.5. That is, given a matrix semantics, a conditional is *natural* if the following conditions are fulfilled. (1) It coincides with the classical conditional when restricted to the classical values T and F; (2) it satisfies *Modus Ponens*; and (3) it is assigned a designated value whenever the antecedent and the consequent are assigned the same value. The original definition in [41] differs from the one just given in condition (3), which reads in [41] as follows: (3) it is assigned a designated value whenever the value assigned to the antecedent is less than or equal to the value assigned to the consequent. Let us briefly comment the motivation behind our weakening of Tomova's definition.

Traditionally, the conditional (or "implication", as it is named in [43], cf. pp. 277, ff.) is required to meet the following restriction of Tomova's condition (3): (3′) $a \to b = t$ iff $a \leq b$, where t is the greatest element in the set of logical values, no other designated elements being considered in this set (cf., e.g., [43], p. 277; or [28], pp. 179, ff.). But property (3′) is only predicable of strong logics. For instance, a logic needs to have the rule "verum e quodlibet" (veq), "If A, then $B \to A$", in order to fulfill (3′) (cf. [43], pp. 227-228; "verum e quodlibet" means "A true proposition follows from any proposition"). For instance, only 4 of the 108 implicative expansions of MK3 considered in this paper meet condition (3′) (these are the ones built upon tables $t9_0$, $t9_1$, $t12_0$ and $t12_1$ in Appendix II).

In this sense, Tomova's notion of a natural conditional can be viewed as an attempt at extending the class of implications worthy of the name beyond the restrictive limits imposed by condition (3′). And certainly, there are very interesting implicative expansions of MK3 satisfying Tomova's definition, such as, for example, the ones determining the paraconsistent logic Pac (cf. [20]), the quasi-relevant logic RM3 (cf. [9]) or the intermediate logic with strong negation named "the logic of the three-element algebra" in [23] (cf. tables $t21_1$, $t24_1$ in Appendix I and $t27_1$ in Appendix II).

Nevertheless, it has to be remarked that an important family of non-classical logics, relevant logics, is not covered by Tomova's definition: none of the implicative expansion of MK3 complying with Tomova's notion of a natural conditional is free from paradoxes of relevance (cf. Appendix III). In this respect, to extend the set of natural conditionals in order to include logics without paradoxes of relevance can be seen as one of the motivations behind the weakening of condition (3) in Tomova's definition. Other properties of the elements we introduce in the family of natural conditionals remain to be investigated, but one of them is clear, as it follows from the results recorded in the following pages: their functional strength, which is precisely the aspect of the logics here introduced we are interested in in the present paper.

Returning to the relevant (i.e., without paradoxes of relevance) logics defined in the sections to follow, let us note a couple of remarks. Firstly, notice that the condition (3) proposed by us allows to assign a non-designated value to a conditional $a \to b$ such that $a \leq b$, not only when a and b are designated values, but also when a is a not designated one. These features frequently appear in matrices for relevant logics, as exemplified in Belnap's matrix M_0 (cf. [6], [1]), a matrix of great importance in the development of relevant logic (cf. [36]), for instance. Thus, we have defined 11 logics with the "variable-sharing property" and 4 more with a related strong property we have named "quasi variable-sharing property". It has to be noted that none of these logics is included in the quasi-relevant logic RM3, considered as the strongest member in the (*latto sensu*) relevant logic family (cf. Appendix III). Secondly, it has to be recalled that there are some many-valued logics with the "variable-sharing property". For example, the aforementioned Belnap's matrix M_0 and Meyer's Crystal lattice CL are axiomatized in [10]. But the meaning of the logical values in these matrices is not easily interpreted from an intuitive point of view. However, the meaning of the three values in the implicative expansions of MK3 is crystalline, for instance, as interpreted in Belnap and Dunn's "bivalent" semantics (cf. [7], [8], [12], [13]).

The aim of this paper is twofold. (1) To prove that the logics determined by all natural implicative expansions of MK3 with only one designated value are functionally equivalent to Łukasiewicz's 3-valued logic Ł3 (cf. [24]). (2) To prove that the logics determined by a certain class of natural implicative expansions of MK3 with two designated values are functionally equivalent to the paraconsistent logic J3 (cf., e.g., [11], pp. 18, ff.). (A more precise account of this aim is provided at the end of the following section.)

2 Preliminary notions

In this section, we record some preliminary notions as used in the present paper (of course, there are alternative definitions of these notions).

Definition 2.1 (Language) The propositional language consists of a denumerable set of propositional variables $p_0, p_1, ..., p_n, ..$, and the following connectives $\to$ (conditional), $\wedge$ (conjunction), $\vee$ (disjunction), $\neg$ (negation). The biconditional ($\leftrightarrow$) and the set of wffs are defined in the customary way. A, B etc. are metalinguistic variables. Sometimes we shall refer by $\mathcal{P}$ and $\mathcal{F}$ to the set of all propositional variables and all formulas, respectively.

Definition 2.2 (Logical matrix) A (logical) matrix is a structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V}$ is a (ordered) set of (truth) values; (2) D is a non-empty proper subset of $\mathcal{V}$ (the set of designated values); and (3) $\mathtt{F}$ is the set of n-ary functions on $\mathcal{V}$ such that for each n-ary connective c (of the propositional language in question), there is a function $f_c \in \mathtt{F}$ such that $\mathcal{V}^n \to \mathcal{V}$. An M-interpretation is a function from $\mathcal{F}$ to $\mathcal{V}$ according to the functions in $\mathtt{F}$.

In this paper logics are *prima facie* considered as structures determined by matrices. In particular, logics are defined as follows.

Definition 2.3 (Logics) Given a matrix M, a logic LM is a structure ($\mathcal{L}$, $\vDash_{\mathrm{M}}$) where $\mathcal{L}$ is a propositional language and $\vDash_{\mathrm{M}}$ is a (consequence) relation defined on $\mathcal{L}$ according to M as follows: for any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ iff $I(A) \in D$ whenever $I(\Gamma) \in D$ for all M-interpretations I ($I(\Gamma) \in D$ iff $I(A) \in D$ for all $A \in \Gamma$). In particular, $\vDash_{\mathrm{M}} A$ (A is M-valid) iff $I(A) \in D$ for all M-interpretations I.

Next, Kleene's strong 3-valued matrix is defined. Notice that we can choose only 2 or else 1 and 2 as designated values. (Kleene uses 1/2 and 1 instead of 1 and 2, respectively. The latter have been chosen in order to use the tester in [16], in case the reader needs one. Also, to put in connection the results in the present paper with previous and subsequent work by us —cf. the concluding remarks in section 8.)

Definition 2.4 (Kleene's strong 3-valued matrix) The propositional language consists of the connectives $\wedge, \vee, \neg$. Kleene's strong 3-valued matrix, MK3 (our label), is the structure $(\mathcal{V}, D, \mathtt{F})$ where (1) $\mathcal{V} = \{0, 1, 2\}$ and it is ordered as shown in the following lattice

(2) $D = \{1, 2\}$ or $D = \{2\}$; (3) $\mathtt{F} = \{f_\wedge, f_\vee, f_\neg\}$ where $f_\wedge$ and $f_\vee$ are defined as the glb (or lattice meet) and the lub (or lattice joint), respectively, and $f_\neg$ is an involution with $f_\neg(2) = 0, f_\neg(0) = 2$ and $f_\neg(1) = 1$. We display the tables for $\wedge$, $\vee$ and $\neg$:

$\wedge$	0	1	2
0	0	0	0
1	0	1	1
2	0	1	2

$\vee$	0	1	2
0	0	1	2
1	1	1	2
2	2	2	2

$\neg$	0
0	2
1	1
2	0

The notion of an MK3-interpretation is defined according to the general Definition 2.2.

The logic determined by MK3 can be named here $K3^1$ (only one designated value) or $K3^2$ (two designated values) (cf. [13], §2.4 on these logics). Then, all logics defined in this paper are natural implicative expansions of either $K3^1$ or else $K3^2$.

In what follows, natural conditionals and the natural implicative expansions of MK3 considered in this paper are defined (there are stricter definitions of a natural conditional; cf. e.g., [3]). Following Tomova [41], we define "natural conditionals" as follows.

Definition 2.5 (Natural conditionals) Let $\mathcal{V}$ and D be defined as in Definition 2.4. Then, an $f_\rightarrow$-function on $\mathcal{V}$ defines a natural conditional if the following conditions are satisfied:

1. $f_\rightarrow$ coincides with (the $f_\rightarrow$-function for) the classical conditional when restricted to the subset $\{0, 2\}$ of $\mathcal{V}$.
2. $f_\rightarrow$ satisfies Modus ponens, that is, for any $a, b \in \mathcal{V}$, if $a \rightarrow b \in D$ and $a \in D$, then $b \in D$.
3. For any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a = b$.

Remark 2.6 (Natural conditionals in Tomova's original paper) We note that natural conditionals are defined in [41] exactly as in Definition 2.5 except for condition (3), which reads there as follows: For any $a, b \in \mathcal{V}$, $a \rightarrow b \in D$ if $a \leq b$.

Proposition 2.7 (Natural conditionals in 3-valued matrices) *(a) Two designated values: Let $\mathcal{L}$ be a propositional language and M be a 3-valued matrix where $\mathcal{V}$ is defined exactly as in MK3 and $D = \{1, 2\}$. Now, consider the 54 $f_\rightarrow$ functions defined in the following general table, where a_i $(1 \leq i \leq 3) \in \{0, 1, 2\}$ and $b_1 \in \{1, 2\}$:*

TI

$\rightarrow$	0	1	2
0	2	a_1	2
1	0	b_1	a_2
2	0	a_3	2

The set of functions contained in TI is the set of all natural conditionals definable in M.

(b) One designated value: Now, let M and $\mathcal{V}$ be as above but $D = \{2\}$. Consider the $f_\rightarrow$ functions in the following general table, where a_i $(1 \leq i \leq 3) \in \{0, 1, 2\}$ and $b_1 \in \{0, 1\}$:

TII

$\rightarrow$	0	1	2
0	2	a_1	2
1	a_2	2	a_3
2	0	b_1	2

The set of 54 functions contained in TII is the set of all natural conditionals definable in M.

Proof It is obvious.

The 54 tables in TI are displayed in Appendix I. They are divided in two groups: 27 tables in TIII and 27 tables in TIV. For n $(1 \leq n \leq 27)$, tables in TIII (resp., TIV) are referred to by tn_2 (resp., tn_1), since $f_{\to}(1,1) = 2$ (resp., $f_{\to}(1,1) = 1$). (Tables TIII and TIV are displayed below, at the end of the section).

The 54 tables in TII are displayed in Appendix II. For n $(1 \leq n \leq 27)$, tables in TII are referred to by tn_0 (resp., tn_1) if $f_{\to}(2,1) = 0$ (resp., $f_{\to}(2,1) = 1$).

Concerning TII, all tables are new except t9_0, t9_1, t18_0, t18_1, t27_0 and t27_1, which appear in Tomova's table TII′ recorded below. Concerning TI, the 30 tables t1_i through t10_i, t13_i, t16_i, t19_i, t22_i and t25_i (with $i \in \{1,2\}$) are the new tables introduced by us, the rest of the tables in TI also appearing in Tomova's table TI′ defined below.

Remark 2.8 (Natural conditionals in 3-valued matrices II) If the original definition of a natural conditional in [41] is adopted instead of Definition 2.5, then the set of natural conditionals in 3-valued matrices is as follows.

(a) Two designated values: The set of 24 functions contained in Table I′

TI′

$\to$	0	1	2
0	2	a_1	2
1	0	a_2	a_3
2	0	b_1	2

where a_i $(1 \leq i \leq 3) \in \{1,2\}$ and $b_1 \in \{0,1,2\}$.

(b) One designated value: The set of 6 functions contained in Table II′

TII′

$\to$	0	1	2
0	2	2	2
1	a	2	2
2	0	b	2

where $a \in \{0,1,2\}$ and $b \in \{0,1\}$.

It is clear that all tables in TI′ (resp., TII′) are contained in TI (resp., TII), although, of course, not conversely.

Tomova's 24 tables in TI′ are t11_i, t12_i, t14_i, t15_i, t17_i, t18_i, t20_i, t21_i, t23_i, t24_i, t26_i, t27_i (with $i \in \{1,2\}$), displayed in Appendix I. As pointed out above, Tomova's 6 tables in TII′ are t9_0, t9_1, t18_0, t18_1, t27_0 and t27_1, which appear in Appendix II.

Definition 2.9 (Natural implicative expansions of MK3) Consider the matrix MK3 and let M be an expansion of it built by adding any $f_{\to}$-function. It is said that M is an implicative expansion of MK3. Then, M is a natural implicative expansion of MK3 if $f_{\to}$ is any of the $f_{\to}$-functions (defining one of the conditionals) in TI or TII (in Proposition 2.7).

We shall use the following referential convention.

Definition 2.10 (T-logics) By a Tx-logic ($x \in \{I, II, III, IV\}$) we mean the logic LM, M being a natural implicative expansion of MK3 built from a $f_{\rightarrow}$-function in table Tx. That is, by a Tx-logic, we refer to the logic determined by the natural implicative expansion of MK3 defined by an $f_{\rightarrow}$-function in Tx. (Tables TIII and TIV are displayed below.)

Next the notions of "propositional connectives definable in a logic" and "functionally equivalent logics" are defined.

Definition 2.11 (Connectives definable in a logic) Let L be a logic determined by the matrix M (cf. Definition 2.3). An n-ary connective c is definable in L iff there is a formula A in L in which only connectives of L occur and at most the propositional variables $p_1, ..., p_n$ and such that for any M-interpretation I, $I(c(p_1, ..., p_n) \leftrightarrow A) \in D$.

Definition 2.12 (Functional inclusion) The logic L is functionally included in the logic L′ iff every primitive or definable connective of L is a primitive or definable connective of L′.

Definition 2.13 (Functional equivalence) The logics L and L′ are functionally equivalent iff L is functionally included in L′ and L′ is functionally included in L.

To end the section, let us review the aims of the paper stated in its introduction. It will be proved:

1. The 54 TII-logics are functionally equivalent to Łukasiewicz's 3-valued logic Ł3 (cf. [24]).
2. Consider the general table TIII contained in Table I:

TIII

$\rightarrow$	0	1	2
0	2	a_1	2
1	0	2	a_2
2	0	a_3	2

   where a_i $(1 \leq i \leq 3) \in \{0, 1, 2\}$. The 27 TIII-logics are functionally equivalent to the paraconsistent logic J3 (cf., e.g., [11] or [20]).
3. Finally, consider the general table TIV contained in Table I:

TIV

$\rightarrow$	0	1	2
0	2	a_1	2
1	0	1	a_2
2	0	a_3	2

   where a_i $(1 \leq i \leq 3) \in \{0, 1, 2\}$. We have (a) the 27 TIV-logics are functionally included in any of the TIII-logics; (b) the converse of (a), however, does not hold: none of the TIII-logics is definable from the TIV-logics; (c) the TIV-logics are not in general functionally equivalent to each other.

As pointed out above, all Tomova's tables with two designated values appear either in TIII or else in TIV (cf. Appendix I).

Aim (1) is fulfilled in sections 3 and 4, while aims (2) and (3) are met in sections 5, 6 and 7. In section 8, we point out some remarks on the results obtained and some suggestions for further work that might enhance the results here obtained. We include two appendices displaying the general tables TI and TII (tables in TII are listed in Appendix I; tables in TIII and TIV, in Appendix II). In Appendix III, we study the behaviour of the 108 logics considered in this paper w.r.t. the "variable-sharing property" and related properties.

3 The 6 tables in TII′ are definable from each other

Let M be a natural implicative expansion of MK3 (cf. Definition 2.9). As remarked above (cf. Definition 2.3), by LM, we refer to the logic determined by M. Then, for simplicity reasons, we proceed as follows. Let M and M′ be two natural implicative expansions of MK3. We show that LM is functionally included in LM′ by proving that the $f_{\rightarrow}$-function in M is definable in M′. Consequently, the proofs of definability of some table by another given in the following pages have to be understood in this sense and as given in the context of two or more natural implicative expansions of MK3. In this and the following section, by tn_0, tn_1, or simply, tn ($1 \leq n \leq 54$), we refer to the tables in Appendix II.

In this section we prove that the 6 tables in TII′ (cf. Remark 2.8) are definable from each other. This fact is proved by Tomova in [40] (cf. also [39], [41], [42] and [21]). A new proof is provided below for both keeping the paper self-contained and using some of the tools employed in our proof in subsequent sections.

For the reader's convenience we write the six tables below

tŁ3.

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	2
2	0	1	2

tŁ3$_0$.

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	2
2	0	0	2

tG3.

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	1	2

tS5.

$\rightarrow$	0	1	2
0	2	2	2
1	0	2	2
2	0	0	2

tC$_0$.

$\rightarrow$	0	1	2
0	2	2	2
1	2	2	2
2	0	0	2

tC$_1$.

$\rightarrow$	0	1	2
0	2	2	2
1	2	2	2
2	0	1	2

The labels of the tables are explained as follows: tŁ3$_0$ is Ł3-implicative table tŁ3 when $b_1 = 0$; C refers to the fact that tC$_0$ and tC$_1$ verify the axioms and rule of classical implicative propositional logic as firstly defined in [25] (C$_0$ and C$_1$ indicate that $b_1 = 0$ and $b_1 = 1$, respectively); G3 refers to Gödel's 3-valued logic G3, the conditional of which is given by tG3 (cf. [5] on Gödel's logics and [29] on 3-valued Gödel logic G3); and, finally, S5 refers to the fact that tS5 verify the axioms and rule of the positive fragment of Lewis' S5 as axiomatized by Hacking [18].

Next, we proceed into proving the interdefinability of the six tables. But before we need a preliminary lemma.

Lemma 3.1 (Additional negations) *Consider any expansion of MK3 by addition of any of the 729 $f_\to$-functions defined in the following general truth-table:*

$\to$	0	1	2
0	e_1	e_2	2
1	e_3	2	e_4
2	0	e_5	e_6

where e_i $(1 \leq i \leq 6) \in \{0, 1, 2\}$. Then, the additional negation connectives $\overset{\bullet}{\neg}$ and $\overset{\circ}{\neg}$ given by the truth-tables

	$\overset{\bullet}{\neg}$
0	2
1	2
2	0

	$\overset{\circ}{\neg}$
0	2
1	0
2	0

are definable.

Proof Set, for any wff A, $\overset{\bullet}{\neg}A = A \to \neg A$ and $\overset{\circ}{\neg}A = \neg(\neg A \to A)$.

Remark 3.2 (On the additional negations) The negation connectives $\neg$, $\overset{\bullet}{\neg}$ and $\overset{\circ}{\neg}$ are instances of the three main negation functions considered in [17] (Section 5.2). That is, Łukasiewicz-type negation, Gödel-type negation, and "dual Gödel-type negation", which are, respectively:

$$\mathrm{non}_1(u) =_{df} 1 - u$$
$$\mathrm{non}_0(u) =_{df} 1, \text{ if } u = 0; \ 0, \text{ otherwise}$$
$$\mathrm{non}^*(u) =_{df} 1, \text{ if } u < 1; \ 0 \text{ if } u = 1$$

Gottwald notes ([17], p. 85) that non_0 and non^* are "extremes examples in the sense that for each negation function n it holds $\mathrm{non}_0 \leq n \leq \mathrm{non}^*$".

Then, we have:

Theorem 3.3 (Interdefinability of the six tables in TII′) *The six tables in TII′ are interdefinable from each other.*

Proof We note the following preliminary remarks: by $\overset{\mathrm{G3}}{\to}$ we refer to the conditional defined by tG3, other superscripts being read similarly; we use the additional negations defined in Lemma 3.1; in the proof to follow, A and B refer to any wffs; in case a tester is needed, the reader can use that in [16]. We have

1. tG3 $\Rightarrow$ tŁ3: $A \overset{\text{Ł3}}{\to} B =_{df} \neg A \vee (A \overset{\mathrm{G3}}{\to} B)$

That is, the conditional $A \overset{\text{Ł3}}{\rightarrow} B$, characteristic of Łukasiewicz's 3-valued logic Ł3 and defined by the table tŁ3, is definable by tG3 with the scheme $\neg A \vee (A \overset{\text{G3}}{\rightarrow} B)$. The items to follow are read similarly.

2. tŁ3 ⇒ tS5: $A \overset{\text{S5}}{\rightarrow} B =_{df} (\overset{\bullet}{\neg}B \overset{\text{Ł3}}{\rightarrow} \overset{\bullet}{\neg}A) \wedge (\overset{\circ}{\neg}B \overset{\text{Ł3}}{\rightarrow} \overset{\circ}{\neg}A)$

3. tS5 ⇒ tG3: $A \overset{\text{G3}}{\rightarrow} B =_{df} B \vee (A \overset{\text{S5}}{\rightarrow} B)$

Therefore, by 1, 2 and 3, we have:

4. tG3 ⇔ tŁ3 ⇔ tS5

That is, tG3, tŁ3 and TS5 are definable from each other. Similar expressions to follow are read similarly.

5. tŁ3 ⇒ tŁ3_0: $A \overset{\text{Ł3}_0}{\rightarrow} B =_{df} \overset{\bullet}{\neg}\overset{\bullet}{\neg}B \vee [\overset{\bullet}{\neg}A \wedge (A \overset{\text{Ł3}}{\rightarrow} B)]$

6. tŁ3_0 ⇒ tŁ3: $A \overset{\text{Ł3}}{\rightarrow} B =_{df} B \vee (A \overset{\text{Ł3}_0}{\rightarrow} B)$

So, by 4, 5 and 6, we obtain

7. tG3 ⇔ tŁ3 ⇔ tŁ3_0 ⇔ tS5

Then, we prove

8. tC_1 ⇒ tC_0: $A \overset{\text{C}_0}{\rightarrow} B =_{df} \overset{\bullet}{\neg}\overset{\bullet}{\neg}B \vee [\overset{\bullet}{\neg}A \wedge (A \overset{\text{C}_1}{\rightarrow} B)]$

9. tC_0 ⇒ tC_1: $A \overset{\text{C}_1}{\rightarrow} B =_{df} B \vee (A \overset{\text{C}_0}{\rightarrow} B)$

10. tG3 ⇒ tC_1: $A \overset{\text{C}_1}{\rightarrow} B =_{df} \overset{\bullet}{\neg}A \vee (A \overset{\text{G3}}{\rightarrow} B)$

11. tC_1 ⇒ tS5: $A \overset{\text{S5}}{\rightarrow} B =_{df} (\overset{\bullet}{\neg}B \overset{\text{C}_1}{\rightarrow} \overset{\bullet}{\neg}A) \wedge (\overset{\circ}{\neg}B \overset{\text{C}_1}{\rightarrow} \overset{\circ}{\neg}A)$

So, finally, by 7, 8, 9, 10 and 11, we have:

12. tG3 ⇔ tŁ3 ⇔ tS5 ⇔ tŁ3_0 ⇔ tC_1 ⇔ tC_0

as was to be proved.

We note that the proof given above could have been slightly simplified by using the precompleteness of Ł3 (cf. [14], where this property of Ł3 was firstly proved; cf. also [4]).

4 The 54 tables in TII are definable from each other

We prove that the 54 tables in TII are definable from each other. As pointed out above, by tn_0, tn_1, or simply, tn ($1 \leq n \leq 54$), we refer to the tables in Appendix II. We begin by proving the following proposition.

Proposition 4.1 (The 27 tables in TII with $b_1 = 0$ are definable by t9)
The 27 tables in TII with $b_1 = 0$ are definable by t9 (tS5).

Proof Once remarks similar to those at the beginning of the proof of Theorem 3.3 are made, we have:

1. t9, t18 and t27 are tS5, tŁ3_0 and tC_0, respectively, which are interdefinable by Theorem 3.3.
2. Secondly, we set the following definition:

$$\text{t3: } A \overset{\text{t3}}{\rightarrow} B =_{df} \overset{\bullet\bullet}{\neg\neg}B \vee (A \overset{\text{S5}}{\leftrightarrow} B)$$

That is, table t3 is definable by means of the scheme $\overset{\bullet\bullet}{\neg\neg}B \vee (A \overset{\text{S5}}{\leftrightarrow} B)$. The items to follow are read similarly.

$$\text{t7: } A \overset{\text{t7}}{\rightarrow} B =_{df} \overset{\circ}{\neg}A \vee (A \overset{\text{S5}}{\leftrightarrow} B)$$

$$\text{t12: } A \overset{\text{t12}}{\rightarrow} B =_{df} (\neg A \wedge \overset{\circ}{\neg}B) \vee [\overset{\bullet\bullet}{\neg\neg}B \vee (A \overset{\text{S5}}{\leftrightarrow} B)]$$

$$\text{t15: } A \overset{\text{t15}}{\rightarrow} B =_{df} \overset{\bullet\bullet}{\neg\neg}B \vee (A \overset{\text{Ł}3_0}{\leftrightarrow} B)$$

$$\text{t16: } A \overset{\text{t16}}{\rightarrow} B =_{df} \overset{\circ}{\neg}A \vee (A \overset{\text{Ł}3_0}{\leftrightarrow} B)$$

$$\text{t17: } A \overset{\text{t17}}{\rightarrow} B =_{df} \neg A \vee (A \overset{\text{S5}}{\leftrightarrow} B)$$

$$\text{t21: } A \overset{\text{t21}}{\rightarrow} B =_{df} (\overset{\bullet}{\neg}A \wedge \overset{\circ}{\neg}B) \vee [\overset{\bullet\bullet}{\neg\neg}B \vee (A \overset{\text{S5}}{\leftrightarrow} B)]$$

$$\text{t25: } A \overset{\text{t25}}{\rightarrow} B =_{df} \overset{\circ}{\neg}A \vee (A \overset{\text{C}_0}{\leftrightarrow} B)$$

$$\text{t26: } A \overset{\text{t26}}{\rightarrow} B =_{df} \neg A \vee (A \overset{\text{C}_0}{\leftrightarrow} B)$$

3. Finally, the rest of the tables are defined by either union ($\cup$) or intersection ($\cap$) of the tables previously defined. (The union (intersection) of two tables is defined by the disjunction (conjunction) of the respective schemes defining them). We have:

t2: t3 $\cap$ t17
t1: t2 $\cap$ t7
t8: t2 $\cup$ t7
t5: t8 $\cap$ t15
t4: t5 $\cap$ t7
t6: t5 $\cup$ t3
t10: t12 $\cap$ t16
t13: t15 $\cap$ t16
t14: t2 $\cup$ t13
t11: t12 $\cap$ t14
t19: t21 $\cap$ t25
t20: t21 $\cap$ t26
t22: t13 $\cup$ t19
t23: t13 $\cup$ t20
t24: t15 $\cup$ t21

For example, consider t2: t3 $\cap$ t17. t3_0 and t17_0 are

t3_0.

$\rightarrow$	0	1	2
0	2	0	2
1	0	2	2
*2	0	0	2

t17_0.

$\rightarrow$	0	1	2
0	2	2	2
1	1	2	1
*2	0	0	2

Now, by using the $f_\wedge$-function in MK3 we have $f_\wedge(2,2) = 2, f_\wedge(0,2) = 0, f_\wedge(2,2) = 2, f_\wedge(0,1) = 0, f_\wedge(2,2) = 2, f_\wedge(2,1) = 1, f_\wedge(0,0) = 0, f_\wedge(0,0) =$

0 and $f_{\wedge}(2,2) = 2$. That is, the table corresponding to $(A \overset{t3_0}{\to} B) \wedge (A \overset{t17_0}{\to} B)$ is:

t2_0.

$\to$	0	1	2
0	2	0	2
1	0	2	1
*2	0	0	2

By leaning on Proposition 4.1, we can prove:

Proposition 4.2 (The 27 tables in TII with $b_1 = 1$ are definable by t9)
The 27 tables in TII with $b_1 = 1$ are definable by t9 (tS5).

Proof Consider the table $t_{\overset{\bullet}{\wedge}}$

$\overset{\bullet}{\wedge}$	0	1	2
0	0	0	0
1	0	0	0
2	0	1	2

defined by the scheme $(A \wedge B) \wedge \overset{\bullet}{\neg}\overset{\bullet}{\neg}A$. And let t_{no} be any of the 27 tables with $b_1 = 0$. Then, $tn_0 \cup t_{\overset{\bullet}{\wedge}} = tn_1$; that is, $tn_0 \cup t_{\overset{\bullet}{\wedge}}$ defines a table that is exactly as tn_0 save for $f_{\to}(2,1)$, which is 1 instead of 0 ($b_1 = 1$ instead of $b_1 = 0$). Thus, given any of the 27 tables with $b_1 = 0$, the corresponding table with $b_1 = 1$ is, as shown, immediately defined (we note that $tS5_1$ (= tG3), tŁ3 and tC_1 are $t9_1$, $t18_1$, and $t27_1$, respectively).

By the propositions just proved, we have the following:

Corollary 4.3 (The tables in TII are definable from the tables in TII′) *The 54 tables in TII are definable from any of the tables in TII′.*

Proof Immediate by Theorem 3.3, Proposition 4.1 and Proposition 4.2.

On the other hand, we prove:

Proposition 4.4 (The tables in TII′ are definable from those in TII)
The six tables in TII′ are definable from any of the tables in TII.

Proof By Theorem 3.3, it suffices to prove that t9 (tS5) is definable by any of the 54 tables in TII. Let tn be any of these 54 tables. Then, t9 (tS5) is definable as follows

$$A \overset{S5}{\to} B =_{df} (\overset{\bullet}{\neg}B \overset{tn}{\to} \overset{\bullet}{\neg}A) \wedge (\overset{\circ}{\neg}B \overset{tn}{\to} \overset{\circ}{\neg}A)$$

Therefore, we conclude:

Theorem 4.5 (Interdefinability of the 54 tables in TII) *The 54 tables in TII are definable from each other.*

Proof Immediate by Corollary 4.3 and Proposition 4.4.

Furthermore, we have:

Theorem 4.6 (Functional equivalence of the 54 TII-logics) *The 54 TII-logics are functionally equivalent to Łukasiewicz logic Ł3.*

Proof Immediate by Theorem 4.5, as the 54 $f_{\rightarrow}$-functions are interdefinable, given the matrix MK3, and one of these functions defines the conditional of Ł3.

5 The 27 tables in TIII are definable from each other

We prove that the 27 tables in TIII (cf. Section 2) are definable from each other. By tn ($1 \leq n \leq 27$), we refer to the 27 first tables in Appendix I. Consider the following tables:

ta.

$\circledcirc$	0	1	2
0	2	0	0
1	0	2	0
2	0	2	2

tb.

$\odot$	0	1	2
0	2	1	0
1	0	2	0
2	0	0	2

tc.

$\overset{\bullet}{\wedge}$	0	1	2
0	0	0	0
1	0	0	0
2	0	1	2

Table tc is used un Proposition 4.2 above. Concerning tables ta and tb, we prove:

Lemma 5.1 (Auxiliary tables ta and tb) *The auxiliary tables ta and tb are definable by t21.*

Proof We follow the conventions used on Sections 3 and 4 above. Firstly, we prove that t24 is definable by t21.

$$\text{t24: } A \overset{\text{t24}}{\rightarrow} B =_{df} B \vee (A \overset{\text{t21}}{\rightarrow} B)$$

That is, the conditional $A \overset{\text{t24}}{\rightarrow} B$ defined in table t24 is definable with the scheme $B \vee (A \overset{\text{t21}}{\rightarrow} B)$ by t21. The items to follow along this section are read similarly. Tables ta and tb are defined as follows:

$$\text{ta: } A \circledcirc B =_{df} (\overset{\bullet}{\neg}\overset{\bullet}{\neg}A \wedge \overset{\circ}{\neg}\overset{\circ}{\neg}B) \vee (A \overset{\text{t21}}{\leftrightarrow} B)$$

$$\text{tb: } A \odot B =_{df} [\overset{\circ}{\neg}A \vee (A \overset{\text{t21}}{\leftrightarrow} B)] \wedge [\neg B \vee (A \overset{\text{t24}}{\leftrightarrow} B)]$$

Then, we can prove the following proposition.

Proposition 5.2 (The 27 tables in TIII are definable by t21) *The 27 tables in TIII are definable by t21.*

Proof

1. Firstly, we set the following definitions:

$$\text{t3: } A \overset{\text{t3}}{\to} B =_{df} \overset{\bullet}{\neg}\overset{\bullet}{\neg}B \vee (A \overset{\text{t21}}{\leftrightarrow} B)$$

$$\text{t6: } A \overset{\text{t6}}{\to} B =_{df} \overset{\bullet}{\neg}\overset{\bullet}{\neg}B \vee (A \overset{\text{t24}}{\leftrightarrow} B)$$

$$\text{t15: } A \overset{\text{t15}}{\to} B =_{df} B \vee (A \overset{\text{t21}}{\leftrightarrow} B)$$

$$\text{t19: } A \overset{\text{t19}}{\to} B =_{df} \overset{\circ}{\neg}A \vee (A \overset{\text{t21}}{\leftrightarrow} B)$$

$$\text{t23: } A \overset{\text{t23}}{\to} B =_{df} \overset{\circ}{\neg}A \vee (A \overset{\text{t24}}{\leftrightarrow} B)$$

$$\text{t27: } A \overset{\text{t27}}{\to} B =_{df} \overset{\circ}{\neg}\overset{\circ}{\neg}B \vee (A \overset{\text{t21}}{\leftrightarrow} B)$$

(Notice that the proofs for t3 and t19 already appeared in Proposition 4.1.)
2. Next, the rest of the tables are defined by either union (∪) or intersection (∩) of the tables previously defined. (The union (intersection) of two given tables is defined by the disjunction (conjunction) of the respective schemes defining them; cf. Proposition 4.1). We have:

t1: t3 ∩ t19
t2: t3 ∩ t23
t4: t1 ∪ tc
t5: t2 ∪ tc
t7: t1 ∪ ta
t8: t2 ∪ ta
t9: t3 ∪ ta
t10: t1 ∪ tb
t11: t2 ∪ tb
t12: t3 ∪ tb
t13: t10 ∪ tc
t14: t11 ∪ tc
t16: t10 ∪ ta
t17: t11 ∪ ta
t18: t12 ∪ ta
t20: t2 ∪ t19
t22: t19 ∪ tc
t25: t19 ∪ ta
t26: t20 ∪ ta

3. Finally, the proof of Proposition 5.2 follows by Lemma 3.1 and Lemma 5.1, (1) and (2).

The converse of Proposition 5.2 is also provable.

Proposition 5.3 (The table t21 is definable from the tables in TIII) *The table t21 is definable from any of the tables in TIII.*

Proof Let tn be any table in TIII. Then, t21 is definable as follows:

$$A \overset{\text{t21}}{\to} B =_{df} (\overset{\bullet}{\neg}B \overset{\text{t}n}{\to} \overset{\bullet}{\neg}A) \wedge (\overset{\circ}{\neg}B \overset{\text{t}n}{\to} \overset{\circ}{\neg}A)$$

Therefore, we conclude:

Theorem 5.4 (Interdefinability of the 27 tables in TIII) *The 27 tables in TIII are definable from each other.*

Proof Immediate by Proposition 5.2 and Proposition 5.3.

Theorem 5.5 (TIII-logics are functionally equivalent) *The 27 TIII-logics are functionally equivalent to each other.*

Proof Immediate by Theorem 5.4.

6 The 27 tables in TIV

The conditional tables upon which some of the most well-known 3-valued paraconsistent logics such as Pac, J3, RM3, Sobociński's, Jaskowski's or Sette's logics (cf. [11], [20], [19], [38], [37] about these logics) are defined do not appear among the 27 tables in TIII. Nevertheless, the referred tables can be defined from any of the tables in TIII. Actually, from each one of the 27 tables in TIII, tn, we easily obtain a corresponding table tn_1 differing from tn only in $f_{\rightarrow}(1,1)$ which equals now 1 instead of 2. In order to prove this fact, we need an auxiliary lemma.

Lemma 6.1 (Auxiliary tables td, te and tf) *Consider the following auxiliary tables*

td.

$\overset{Pac}{\rightarrow}$	0	1	2
0	2	2	2
1	0	1	2
2	0	1	2

te.

$\boxdot$	0	1	2
0	0	0	0
1	0	1	0
2	0	2	2

tf.

$\circledast$	0	1	2
0	2	2	2
1	0	1	2
2	0	2	2

These tables are definable by any of the tables in TIII.

Proof By Theorem 5.4, it suffices to prove that td, te and tf are definable by t21. The required definitions are:

$$A \overset{\text{Pac}}{\rightarrow} B =_{df} \overset{\circ}{\neg} A \vee B$$

$$A \boxdot B =_{df} [(A \vee \overset{\circ}{\neg} B) \wedge \overset{\circ}{\neg}\overset{\circ}{\neg} B] \wedge [\overset{\bullet}{\neg}\overset{\bullet}{\neg} A \vee (A \overset{\text{t21}}{\leftrightarrow} B)]$$

$$A \circledast B =_{df} (A \overset{\text{Pac}}{\rightarrow} B) \vee (A \boxdot B) \text{ (that is, tf = td } \cap \text{ te)}$$

We remark that td is the table upon which the logic Pac is defined (cf. [11], pp. 18, ff. Cf. Definition 7.1 below).

By leaning on Theorem 5.4, we can define the 27 tables with $f_{\rightarrow}(1,1) = 1$.

Proposition 6.2 (The 27 tables in TIV) *For each table tn in TIII, there is a corresponding table tn$_1$ differing from tn only in the value* $f_{\rightarrow}(1,1)$*, which is now* 1 *instead of* 2*. Each one of these new tables is definable from any table in TIII.*

Proof (1) Let tn be a table in TIII with $f_{\rightarrow}(2,1) = 2$. Then, the table tn $\cap$ tf is the required table tn_1. In this way, the corresponding tables to t7-t9, t16-t18 and t25-t27 in TIII are defined. (2) Let tn be a table in TIII with $f_{\rightarrow}(2,1) = 0$ or $f_{\rightarrow}(2,1) = 1$. Then, the table tn $\cap$ td is the required table tn_1. In this way, the corresponding tables to t1-t6, t10-t15 and t19-t24 in TIII are defined. Finally, from Theorem 5.4, it follows that the tables in TIV are definable from any of the tables in TIII.

Concerning the 27 tables with $f_{\rightarrow}(1,1) = 1$, we note the following propositions:

Proposition 6.3 (Tables in TIII and in TIV are not interdefinable) *None of the tables in TIII (where $f_{\to}(1,1) = 2$) is definable from the tables in TIV (where $f_{\to}(1,1) = 1$).*

Proof It is immediate, since for each one of the tables in TIV, we have: $f_{\to}(1,1) = f_{\wedge}(1,1) = f_{\vee}(1,1) = f_{\neg}(1) = 1$. So, $f_{\to}(1,1) = 2$ is not definable.

Proposition 6.4 (Tables in TIV are not interdefinable) *Contrary to what is the case with tables in TIII, tables in TIV are not in general definable from each other.*

Proof Tables t1-t10, t13, t16, t19, t22 and t25 in TIII (cf. Remark 2.8) are not considered by Tomova [41]. The rest of the tables in TIII are treated by her. Among the latter, she remarks t15 and t14 and t12, which are labelled T1, T2 and T3, respectively. Now, in the lattice she defines, T1, T2 and T3 are not equivalent.

7 The tables in TII and TIII and the logics J3 and Ł3

In what follows, we briefly discuss the relations the logics Ł3 and J3 maintain with TII-logics and TIII-logics w.r.t. functional equivalence.

We recall that the 54 TII-logics are functionally equivalent to Ł3 (Theorem 4.6) and that the 27 TIII-logics are functionally equivalent to each other (cf. Theorem 5.5).

Then, we begin by defining the logic J3.

Definition 7.1 (The logics Pac and J3) The logic Pac (Paraconsistent logic) is $\mathrm{LMt24}_1$, that is, the logic determined by the natural implicative expansion, $\mathrm{Mt24}_1$, of MK3. (Notice that t24 (with $f_{\to}(1,1) = 1$) is table td in Lemma 6.1.) Then, the logic J3 is defined by adding the negation $\overset{\circ}{\neg}$ to Pac (cf. Lemma 3.1; cf. [11], pp. 18, ff.)

Now, we prove:

Theorem 7.2 (TIII-logics are functionally equivalent to J3) *The 27 TIII-logics are functionally equivalent to J3.*

Proof As td (i.e., t24 with $f_{\to}(1,1) = 1$) and $\overset{\circ}{\neg}$ are definable from any table in TIII (by Lemma 3.1 and Lemma 6.1), it suffices to prove that, any table in TIII is definable by td and $\overset{\circ}{\neg}$. Now, $A \overset{\mathrm{t27}}{\to} B$ is defined as follows: $A \overset{\mathrm{t27}}{\to} B =_{df} \overset{\circ}{\neg}\overset{\circ}{\neg}B \vee (A \overset{\mathrm{td}}{\to} B)$. So, it is a consequence of Theorem 5.4 that any table in TIII is definable by td and $\overset{\circ}{\neg}$.

On the other hand, it is well known that J3 and Ł3 are functionally equivalent (cf., e.g., [11] or [20] and references therein). But, anyway, let us give here a simple proof of this fact.

Proposition 7.3 (J3 and Ł3 are functionally equivalent) *The logic J3 and Łukasiewicz's 3-valued logic Ł3 are functionally equivalent.*

Proof J3 is functionally included in Ł3: by Lemma 3.1, $\overset{\circ}{\neg}$ is definable; so, by Lemma 6.1, table td (the conditional table of Pac) is also definable. Consequently, J3 is functionally included in Ł3 by Definitions 2.3 and 7.1 (this fact can also be proved by using the precompleteness of Ł3; cf. the remark at the end of section 3). On the other hand, consider now the following table:

■	0	1	2
0	2	0	0
1	1	0	0
2	0	0	0

This table is definable as follows: $A \blacksquare B =_{df} \neg A \wedge \overset{\circ}{\neg} B$. Then, given that all tables in TIII are definable by td and $\overset{\circ}{\neg}$, the Ł3-table is definable as follows: $A \overset{\text{Ł3}}{\to} B =_{df} (A \overset{\text{t24}}{\to} B) \vee (A \blacksquare B)$. So, Ł3 is functionally included in J3.

Finally, we have:

Theorem 7.4 (TII-logics and TIII-logics are functionally equivalent) *The 54 TII-logics and the 27 TIII-logics are functionally equivalent to Ł3.*

Proof Immediate by Theorems 4.6, 5.5, 7.2 and 7.3.

8 Concluding remarks

The paper is ended with some remarks on the results obtained and some suggestions for further work in the same line.

1. In [41], Tomova establishes the lattice defining the relations between natural implicative expansions of not only Kleene's strong logic, but also those extending Kleene's weak logic (cf. [22], §64) and Kleene's "intermediate" logic (cf. [15]). The proofs are referred to her doctoral dissertation [39]. Concerning Kleene's strong logic, she just considers the 30 natural conditionals described in Remark 2.8 above.
2. Most of the logics summarily treated in this paper have not, to our knowledge, been considered previously in the literature. However, there are famous logics between them. Among the tables in Appendix I, we have, for example, the following: $t27_2$ is the conditional table of Sette's P1 (cf. [37], negation is interpreted by $\overset{\circ}{\neg}$); $t21_1$ is the conditional table of Sobociński's S3 (cf. [38]) or the quasi-relevant logic RM3 (cf. [1], [9]). In both cases, negation is interpreted by $\neg$; $t24_1$ is Jaskowski's table for the conditional (cf. [19]). If negation is interpreted by $\neg$, we have the logic Pac, which has appeared in a number of works (perhaps firstly in [2]; cf. [11], pp. 18, ff. or [20] and references therein). Finally, notice that the tables such as those for Priest's LP (cf. [27]) do not appear in Appendices I and II, since it violates condition (2) in Definition 2.5, requiring the verification of Modus Ponens.

3. We remark that Definition 2.5 do not cover the class of all implicative expansions of MK3 determining logics which either functionally include or are functionally equivalent to Ł3. Consider, for example, the following general table ($D = \{1, 2\}$), where $a_i \in \{0, 1, 2\}$ $(1 \leq i \leq 3)$:

TV

$\to$	0	1	2
0	2	a_1	2
1	0	2	a_2
2	0	a_3	1

By using table $\overset{\bullet}{\wedge}$ (cf. Proposition 4.2), we have TV $\cup$ t$\overset{\bullet}{\wedge}$ = TVI, where $a_i \in \{0, 1, 2\}$ $(1 \leq i \leq 2)$ and $b_1 \in \{1, 2\}$.

TVI

$\to$	0	1	2
0	2	a_1	2
1	0	2	a_2
2	0	b_1	2

But TVI is contained in TIII, whence the logics determined by TV functionally include Ł3 (cf. Theorems 5.4, 7.2 and 7.3). So, condition (1) in Definition 2.5 can be weakened. In this sense, it would be interesting to investigate to what extent Definition 2.5 can be weakened while still containing implicative expansions of MK3 determining logics functionally including Ł3 or functionally equivalent to Ł3.
4. Let us generally name TIV′-logics the 15 new logics in TIV introduced by us. By using a Belnap-Dunn bivalent semantics (cf. [7], [8], [12], [13]), Hilbert-style axiomatic systems corresponding to TI′-logics (resp., TII′-logics; resp., TIV′-logics) are defined in [31] (resp., in [35]; resp., in [33]) in a general and unified way (cf. also [30], [32], [34] and [26]). That is, LM being a TI′-logic or a TII′-logic or a TIV′-logic, a proof-theoretical relation $\vdash$ is built in the said papers such that for any set of wffs Γ and wff A, $\Gamma \vDash_{\mathrm{M}} A$ iff $\Gamma \vdash A$ is proved. We wonder whether this result can be extended to all TI-logics and TII-logics.
5. It will be interesting to build the lattice defining the relations between the 27 TIV-logics.

A Appendix I. The 27 tables in TI

The 54 tables in TI are divided in two groups: 27 tables in TIII and 27 tables in TIV. The 27 tables in TIII are the following (designated values are starred). We refer to the tables in this set by tn_2 or simply by tn $(1 \leq n \leq 27)$.

t1.

$\to$	0	1	2
0	2	0	2
*1	0	2	0
*2	0	0	2

t2.

$\to$	0	1	2
0	2	0	2
*1	0	2	1
*2	0	0	2

t3.

$\to$	0	1	2
0	2	0	2
*1	0	2	2
*2	0	0	2

t4.

→	0	1	2
0	2	0	2
*1	0	2	0
*2	0	1	2

t5.

→	0	1	2
0	2	0	2
*1	0	2	1
*2	0	1	2

t6.

→	0	1	2
0	2	0	2
*1	0	2	2
*2	0	1	2

t7.

→	0	1	2
0	2	0	2
*1	0	2	0
*2	0	2	2

t8.

→	0	1	2
0	2	0	2
*1	0	2	1
*2	0	2	2

t9.

→	0	1	2
0	2	0	2
*1	0	2	2
*2	0	2	2

t10.

→	0	1	2
0	2	1	2
*1	0	2	0
*2	0	0	2

t11.

→	0	1	2
0	2	1	2
*1	0	2	1
*2	0	0	2

t12.

→	0	1	2
0	2	1	2
*1	0	2	2
*2	0	0	2

t13.

→	0	1	2
0	2	1	2
*1	0	2	0
*2	0	1	2

t14.

→	0	1	2
0	2	1	2
*1	0	2	1
*2	0	1	2

t15.

→	0	1	2
0	2	1	2
*1	0	2	2
*2	0	1	2

t16.

→	0	1	2
0	2	1	2
*1	0	2	0
*2	0	2	2

t17.

→	0	1	2
0	2	1	2
*1	0	2	1
*2	0	2	2

t18.

→	0	1	2
0	2	1	2
*1	0	2	2
*2	0	2	2

t19.

→	0	1	2
0	2	2	2
*1	0	2	0
*2	0	0	2

t20.

→	0	1	2
0	2	2	2
*1	0	2	1
*2	0	0	2

t21.

→	0	1	2
0	2	2	2
*1	0	2	2
*2	0	0	2

t22.

→	0	1	2
0	2	2	2
*1	0	2	0
*2	0	1	2

t23.

→	0	1	2
0	2	2	2
*1	0	2	1
*2	0	1	2

t24.

→	0	1	2
0	2	2	2
*1	0	2	2
*2	0	1	2

t25.

→	0	1	2
0	2	2	2
*1	0	2	0
*2	0	2	2

t26.

→	0	1	2
0	2	2	2
*1	0	2	1
*2	0	2	2

t27.

→	0	1	2
0	2	2	2
*1	0	2	2
*2	0	2	2

Then, the 27 tables in TIV are obtained by replacing $1 \to 1 = 1$ by $1 \to 1 = 2$ in each one of the tables recorded in the list above. We refer to the tables in this set by tn_1 ($1 \leq n \leq 27$). Thus, for example, T8_1 is:

t8_1.

→	0	1	2
0	2	0	2
*1	0	1	1
*2	0	2	2

B Appendix II. The 54 tables in TII

The 27 tables in which $b_1 = 0$ are the following (designated values are starred). We refer to the tables in this set by tn_0 $(1 \leq n \leq 27)$.

t1.

→	0	1	2
0	2	0	2
1	0	2	0
*2	0	0	2

t2.

→	0	1	2
0	2	0	2
1	0	2	1
*2	0	0	2

t3.

→	0	1	2
0	2	0	2
1	0	2	2
*2	0	0	2

t4.

→	0	1	2
0	2	1	2
1	0	2	0
*2	0	0	2

t5.

→	0	1	2
0	2	1	2
1	0	2	1
*2	0	0	2

t6.

→	0	1	2
0	2	1	2
1	0	2	2
*2	0	0	2

t7.

→	0	1	2
0	2	2	2
1	0	2	0
*2	0	0	2

t8.

→	0	1	2
0	2	2	2
1	0	2	1
*2	0	0	2

t9.

→	0	1	2
0	2	2	2
1	0	2	2
*2	0	0	2

t10.

→	0	1	2
0	2	0	2
1	1	2	0
*2	0	0	2

t11.

→	0	1	2
0	2	0	2
1	1	2	1
*2	0	0	2

t12.

→	0	1	2
0	2	0	2
1	1	2	2
*2	0	0	2

t13.

→	0	1	2
0	2	1	2
1	1	2	0
*2	0	0	2

t14.

→	0	1	2
0	2	1	2
1	1	2	1
*2	0	0	2

t15.

→	0	1	2
0	2	1	2
1	1	2	2
*2	0	0	2

t16.

→	0	1	2
0	2	2	2
1	1	2	0
*2	0	0	2

t17.

→	0	1	2
0	2	2	2
1	1	2	1
*2	0	0	2

t18.

→	0	1	2
0	2	2	2
1	1	2	2
*2	0	0	2

t19.

→	0	1	2
0	2	0	2
1	2	2	0
*2	0	0	2

t20.

→	0	1	2
0	2	0	2
1	2	2	1
*2	0	0	2

t21.

→	0	1	2
0	2	0	2
1	2	2	2
*2	0	0	2

t22.

→	0	1	2
0	2	1	2
1	2	2	0
*2	0	0	2

t23.

→	0	1	2
0	2	1	2
1	2	2	1
*2	0	0	2

t24.

→	0	1	2
0	2	1	2
1	2	2	2
*2	0	0	2

t25.

→	0	1	2
0	2	2	2
1	2	2	0
*2	0	0	2

t26.

→	0	1	2
0	2	2	2
1	2	2	1
*2	0	0	2

t27.

→	0	1	2
0	2	2	2
1	2	2	2
*2	0	0	2

Then, the 27 tables in which $b_1 = 1$ are obtained by replacing $2 \to 1 = 1$ by $2 \to 1 = 0$ in each one of the tables recorded in the list above. We refer to the tables in this set by tn_1 $(1 \leq n \leq 27)$. Thus, for example, t17_1 is:

t17$_1$.

$\to$	0	1	2
0	2	2	2
1	1	2	1
*2	0	1	2

C Appendix III

It is proved that all implicative expansions of MK3 defined by Tomova are not relevant logics in the sense that they lack the "variable-sharing property" (vsp). However, 11 of the implicative expansions of MK3 we have introduced are relevant, as they have the vsp. Also, we examine the behavior of the 108 logics considered in the present paper w.r.t. two vsp-related properties, the "quasi variable-sharing property" (qvsp) and the "weak relevant property" (wrp).

The vsp, the qvsp and the wrp are defined as follows (cf. Definition 2.3).

Definition C.1 (Variable-sharing property —vsp) Let L be a logic defined upon the matrix M. L has the "variable-sharing property" (vsp) if in all M-valid wffs of the form $A \to B$, A and B share at least a propositional variable.

Definition C.2 (Quasi variable-sharing property —qvsp) Let L be a logic defined upon the matrix M. L has the "quasi variable-sharing property" (qvsp) if in all M-valid wffs of the form $A \to B$, A and B share at least a propositional variable or else both A and B are M-valid.

Definition C.3 (Weak relevant property —wrp) Let L be a logic defined upon the matrix M. L has the "weak relevant property" (wrp) if in all M-valid wffs of the form $A \to B$, A and B share at least a propositional variable or else both $\neg A$ and B are M-valid.

To the best of our knowledge, the qvsp is a new property introduced here for the first time. As far as we know, the wrp is first defined in [1] (cf. p. 117), while the vsp is a well-known property (cf. [6], [1] and references in the last item). Nevertheless, notice that the vsp and the wrp are customarily defined leaning upon the notion of L-theoremhood, not upon that of M-validity. Of course, both versions of the vsp and the wrp are equivalent in the presence of a soundness and completeness theorem, as it is the case with the logics mentioned in the first paragraph of this third appendix (cf. conclusion 7 below).

Next, we proceed to the examination of the 108 logics considered in the paper w.r.t. the properties just defined.

Proposition C.4 (TII-logics, the vsp, the qvsp and the wrp) *Let L be a TII-logic- Then, L lacks the vsp, the qvsp and the wrp.*

Proof Immediate since the wff $\neg(p \to p) \to \neg(q \to q)$ is, for any propositional variables p, q, M-valid in any implicative expansion of MK3, M, built upon any table in TII.

Proposition C.5 (TIII-logics, the vsp, the qvsp and the wrp) *Let L be a TIII-logic. Then, L lacks the vsp, the qvsp and the wrp.*

Proof Immediate by using the wff $\neg(p \to p) \to \neg(q \to q)$ similarly as in the precedent proposition.

Thus, concerning the three properties we are interested in in this appendix, we are left with the 27 tables in TIV. In what follows, some facts about TIV-logics w.r.t. these properties are proven. Let us refer by TIV$_i$ ($1 \leq i \leq 27$) to the TIV-logic built upon ti.

Proposition C.6 (TIV-logics and the vsp I) *Let L be the logic TIV$_i$ where* $i \in \{5, 6, 8, 9, 11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27\}$. *Then, L lacks the vsp.*

Proof (a) TIV$_i$ ($i \in \{5, 6, 8, 9\}$). By using the wff $(p \to p) \to (q \to q)$ (p, q are distinct propositional variables) verified by tables t5, t6, t8 and t9.

(b) TIV$_i$ ($i \in \{11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27\}$). By using the wff $\neg(p \to p) \to (q \to q)$ (p, q are distinct propositional variables) verified by tables t11, t12, t14, t15, t17, t18, t20, t21, t23, t24, t26 and t27.

Proposition C.7 (TIV-logics and the qvsp I) *Let L be the logic TIV$_i$ where $i \in \{11, 12, 14, 15, 17, 18, 20, 21, 23, 24, 26, 27\}$. Then, L lacks the qvsp.*

Proof By using the wff $(p\wedge\neg p) \to (q\vee\neg q)$ (p, q are distinct propositional variables) similarly as in the preceding proposition.

Proposition C.8 (TIV-logics and the wrp I) *Let L be the logic TIV$_i$ where $i \in \{5, 6, 8, 9, 14, 15, 17, 18, 23, 24, 26, 27\}$. Then, L lacks the wrp.*

Proof By using the wff $(p \to p) \to (q \to q)$ (p, q are distinct propositional variables) similarly as in Propositions C7 and C8.

On the other hand, we have the following propositions.

Proposition C.9 (TIV-logics and the vsp II) *Let L be the logic TIV$_i$ where $i \in \{1, 2, 3, 4, 7, 10, 13, 16, 19, 22, 25\}$. Then, L has the vsp.*

Proof (a) TIV$_i$ ($i \in \{1, 4, 7, 10, 13, 16, 19, 22, 25\}$). Let M be the implicative expansion of MK3 determining the logic L. Suppose that there are wffs A and B such that $A \to B$ is M-valid but A and B do not share propositional variables. Let I be an M-interpretation assigning 1 (resp., 0) to each propositional variable in A (resp., B). Then $I(A) = 1$ and $I(B) \in \{0, 2\}$, since $\{1\}$ and $\{0, 2\}$ are closed under $\to, \wedge, \vee$ and $\neg$. Consequently, $I(A \to B) = 0$, contradicting the M-validity of $A \to B$.

(b) TIV$_i$ ($i \in \{1, 2, 3\}$). The proof is similar to that of case (a) by using now the fact that $f_\to(0, 1) = f_\to(2, 1) = 0$.

Proposition C.10 (TIV-logics and the qvsp II) *Let L be the logic TIV$_i$ where $i \in \{5, 6, 8, 9\}$. Then, L has the qvsp.*

Proof Let M be the implicative expansion of MK3 determining the logic L. Suppose that there are wffs A and B not having propositional variables in common and such that $A \to B$ is M-valid but either A or B is not.

(a) A is not M-valid: Then, there is an M-interpretation I such that $I(A) = 0$. Let I' be exactly as I except that for each propositional variable p in B, $I'(p) = 1$. Clearly, $I'(B) = 1$ since $\{1\}$ is closed under $\to, \wedge, \vee$ and $\neg$, and $I'(A) = 0$ since A and B do not share propositional variables. Then, $I'(A \to B) = 0$, contradicting the M-validity of A.

(b) B is not M-valid: Then, there is an M-interpretation I such that $I(B) = 0$. Let I' be exactly as I except that for each propositional variable p in A, $I'(p) = 1$. Similarly as in case (a), $I'(A) = 1$, $I'(B) = 0$, and then $I'(A \to B) = 0$, contradicting the M-validity of $A \to B$.

Proposition C.11 (TIV-logics and the wrp II) *Let L be the logic TIV$_i$ where $i \in \{11, 12, 20, 21\}$. Then, L has the wrp.*

Proof Let M be the implicative expansion of MK3 determining the logic L. Suppose that there are wffs A and B not having propositional variables in common and such that $A \to B$ is M-valid but either $\neg A$ or B is not. The proof that, given the suppositions just stated, $A \to B$ is actually not M-valid is similar to that of the preceding proposition leaning now on the fact that $f_\to(1, 0) = f_\to(2, 1) = 0$.

The results we have obtained are summarized below.

1. All TII-logics and TIII-logics lack the vsp, the qvsp and wrp (Propositions C4 and C5).

2. The only TI-logics having the vsp are TIV-logics. In particular, the following 11 TIV-logics: TIV_i where $i \in \{1, 2, 3, 4, 7, 10, 13, 16, 19, 22, 25\}$ (Propositions C4, C5, C6 and C9).
3. The only TI-logics having the qvsp are TIV-logics. In particular, the following 4 TIV-logics, in addition to the 11 TIV-logics having the vsp: TIV_i where $i \in \{5, 6, 8, 9\}$ (Propositions C4, C5, C7 and C10).
4. The only TI-logics having the wrp are TIV-logics. In particular, the following 4 TIV-logics, in addition to the 11 TIV-logics having the vsp: TIV_i where $i \in \{11, 12, 20, 21\}$ (Propositions C5, C6, C8 and C11).
5. Tomova's 30 tables are generally defined in tables TI′ and TII′. The tables with two designated values are also defined in TIII and TIV. Tomova's 24 tables in TIII and TIV are the following: t11, t12, t14, t15, t17, t18, t20, t21, t23, t24, t26 and t27. All these TIII-logics and TIV-logics lack the vsp and the qsvp. Also, all lack the wrp, except TIV_{11}, TIV_{12}, TIV_{20} and TIV_{21} (cf. 1, 2, 3 and 4 above; notice that TIV_{21} is the quasi-relevant logic RM3 —cf. [9], [1] and references therein).
6. The 78 new tables we have introduced together with Tomova's 30 tables are generally defined in tables TI (30 new tables) and TII (48 new tables). The tables with two designated values are also defined in TIII and TIV. These 30 tables are the following: t1 through t9, t10, t13, t16, t19, t22 and t25. Of these, the new 15 TIV-logics are the only members in the set of 108 logics considered in this paper with either the vsp or the qvsp: TIV_i ($i \in \{1, 2, 3, 4, 7, 10, 13, 16, 19, 22, 25\}$) has the vsp; TIV_i ($i \in \{5, 6, 8, 9\}$) has the qvsp.
7. As pointed out in the Concluding remarks (section 8), Hilbert-style axiomatic systems corresponding to Tomova's logics and the 15 new TIV-logics we have introduced are defined in a general and unified way in [31], [35] and [33].

The appendix is ended by remarking that none of the new 15 TIV-logics (with either the vsp or the qvsp) we have introduced is included in RM3, the 3-valued extension of the logic R-Mingle, commonly considered as the strongest logic in the relevant logic family (cf. [3], p. 276). (But notice that RM3 —cf. table TIV_{21}— does not have the vsp.)

Proposition C.12 (The 15 new TIV-logics are not incl. in RM3) *Let L be the logic* TIV_i *where* $i \in \{1, 2, 3, 4, 5, 6, 7, 8, 9, 10, 13, 16, 19, 22, 25\}$*. Then, L is not included in RM3.*

Proof Let L be any of the TIV_i-logics except TIV_3, TIV_6, TIV_9. Then, the rule "If $A \wedge \neg A$, then $\neg(A \rightarrow B)$" holds in L. Next, "If $B \wedge \neg B$, then $\neg(A \rightarrow B)$" holds in TIV_3 and TIV_6 and the thesis $(A \wedge B) \rightarrow (A \rightarrow B)$ holds in TIV_9. However, both aforementioned rules and the thesis just quoted fail in RM3.

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