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Relational Semantics for the Paraconsistent and Paracomplete 4-valued Logic PŁ4

Abstract. The paraconsistent and paracomplete 4-valued logic PŁ4 is originally interpreted with a two-valued Belnap-Dunn semantics. In the present paper, PŁ4 is endowed with both a ternary Routley-Meyer semantics and a binary Routley semantics together with their respective restriction to the 2 set-up case.

Keywords: paraconsistent logics; paracomplete logics; 4-valued logics; modal 4-valued logics; Routley-Meyer semantics; binary Routley semantics; 2 set-up Routley-Meyer semantics; 2 set-up binary Routley semantics

1. Introduction

The logic PŁ4 is a negation expansion of the implicative fragment of classical propositional logic. It is a strong and rich paraconsistent and paracomplete 4-valued logic where necessity and possibility (among other) operators are definable without "Łukasiewicz-type modal paradoxes" being provable [cf. 6, 7, 9]. The logic PŁ4 is defined in [8], but in [5], it is remarked that De and Omori's logic BD_+, Zaitsev's paraconsistent logic FDEP and Beziau's four-valued modal logic PM4M are logics equivalent to PŁ4 [cf. 1, 4, 15]. The fact that the four systems just cited (PŁ4, BD_+, FDEP and PM4M) have been independently obtained from different motivations seems to suggest that they are four versions of a strong and rich natural logic.

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We will briefly recall only some of the properties PŁ4 enjoys (a detailed account of these and other properties of PŁ4 can be consulted in [8]).

1. The logic PŁ4 has the classical deduction theorem, since it contains implicative intuitionistic logic and the sole rule of inference is MP.
2. PŁ4 is self-extensional in the sense that it has the replacement (of equivalents) theorem, as the rule Contraposition is an admissible rule in PŁ4.
3. PŁ4 is a paraconsistent logic in the sense that the rule 'E contradictione quodlibet', Ecq, fails in PŁ4.
4. PŁ4 is a paracomplete logic in the sense that not all prime PŁ4-theories with all PŁ4-theorems contain either A or else $\neg A$, for each formula A.
5. PŁ4 has a great expressive power. For example, normal conjunction and disjunction, necessity and possibility, along with classical, Gödel-type and dual Gödel-type negation operators are definable in PŁ4.
6. Łukasiewicz-type modal paradoxes are not provable in PŁ4.

PŁ4 is originally interpreted with a two-valued Belnap-Dunn semantics [cf. 8] and references therein). The aim of the present paper is to provide still another perspective on PŁ4 by endowing it with both a ternary Routley-Meyer semantics and a binary Routley semantics together with their respective restriction to the 2 set-up case.

Routley-Meyer semantics (RM-semantics), in principle designed for interpreting relevant logics, is nowadays a semantics for non-classical logics in general (cf. [3, 11, 13], and references in these items). Binary Routley semantics (bR-semantics) is introduced in [10] for interpreting expansions of positive intuitionistic logic. It is essentially distinguished from RM-semantics by the accessibility relation defined in the set of all points in the models, which is a binary relation instead of the ternary one characteristic of RM-semantics. 2 set-up Routley-Meyer semantics (2RM-semantics) is appropriate for some 3-valued and 4-valued logics. 2RM-semantics was introduced in [2], but leaving aside [12], the topic has not been pursued, to the best of our knowledge. Finally, 2 set-up binary Routley semantics (2bR-semantics) is going to be introduced in the present paper when PŁ4 is given this kind of semantics.

We remark that the term "set-up" is taken from Routley *et al.* [cf. 13] and references therein), which they use to emphasize the fact that the canonical interpretation of a point in RM-semantics can be an incomplete and/or inconsistent theory. In, say, standard Kripke semantics, the canonical interpretations of the points in the models are complete and consistent theories, as it is known. Routley *et al.* use the term "set-up" in contradistinction to "world", the customary one in Kripke semantics and related types of semantics.

The alternative interpretations of PŁ4 given in the following pages will put it in connection with the wealth of logics which can currently be understood in RM-semantics as well as with the few ones given a 2RM-semantics, while at the same time our knowledge of both relational semantics will be improved.

The paper is organized as follows. In §2, the logic PŁ4 is recalled, and in §3, PŁ4 is given a general RM-semantics and the soundness theorem is proved. In §4, completeness of PŁ4 w.r.t. the semantics introduced in §2 is proved. In §5, 2 set-up RM-semantics for PŁ4 is defined and the soundness and completeness theorems are proved. In §6, PŁ4 is endowed with a bR-semantics and a 2bR-semantics. Finally, in §7, we note some remarks on possible future work to be done on the topic. We have added an appendix presenting some of the connectives definable in PŁ4, as well as the basic positive (i.e., negationless) logics B_+ and B_{K+}, of some interest in the paper.

2. The logic PŁ4

In this section the logic PŁ4 defined in [8] is recalled.

The propositional language consists of a denumerable set of propositional variables $p_0, p_1, \ldots, p_n, \ldots$, and the following connectives: $\rightarrow$ (conditional) and $\neg$ (negation). The set of wffs is defined in the customary way. A, B, C, etc. are metalinguistic variables. PŁ4 is formulated as a Hilbert-type axiomatic system, the notions of 'theorem' and 'proof from a set of premises' being understood in the standard way.

DEFINITION 2.1. The logic PŁ4 can be axiomatized as follows.

Axioms:

A1. $A \rightarrow (B \rightarrow A)$

A2. $[A \rightarrow (B \rightarrow C)] \rightarrow [(A \rightarrow B) \rightarrow (A \rightarrow C)]$

A3. $[(A \rightarrow B) \rightarrow A] \rightarrow A$

A4. $A \rightarrow \neg\neg A$

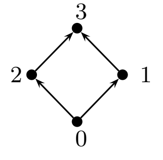
A5. $\neg\neg A \rightarrow A$

A6. $\neg(A \to B) \to (\neg A \to C)$
A7. $\neg(A \to B) \to \neg B$
A8. $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)]$

Rule of inference:

Modus Ponens (MP). $A, A \to B \Rightarrow B$ (if A and $A \to B$, then B)

Definition 2.2 (The matrix MPŁ4). The propositional language consists of the connectives $\to$ and $\neg$. The matrix MPŁ4 is the structure $(\mathcal{V}, D, \mathtt{F})$, where (1) $\mathcal{V}$ is $\{0, 1, 2, 3\}$ and is partially ordered as shown in the following lattice



(2) $D = \{3\}$; $\mathtt{F} = \{f_\to, f_\neg\}$, where $f_\to$ and $f_\neg$ are defined according to the following truth-tables:

$\to$	0	1	2	3	$\neg$
0	3	3	3	3	3
1	2	3	2	3	1
2	1	1	3	3	2
3	0	1	2	3	0

In [8] it is proved that PŁ4 is determined by the degree of truth-preserving consequence relation defined on the ordered set of values of MPŁ4.

Remark 2.1. The following theorems and rule of PŁ4 will be used in the sequel:

(T1) $A \to A$
(T2) $A \to [B \to \neg[(\neg A \to \neg B) \to \neg A]$
(T3) $\neg B \to [\neg A \to \neg[(A \to B) \to B]]$
(T4) $\neg[(\neg(A \to B) \to \neg A) \to \neg A] \to B$
(Efq$_2$) $\vdash_{\text{PŁ4}} A \Rightarrow \vdash_{\text{PŁ4}} \neg A \to B$.

(In the appendix to the paper, we have remarked some connectives definable in PŁ4, as well as some of its conspicuous theorems and rules.)

Remark 2.2. PŁ4 is not a relevant logic: A1 together with T1 and MP provides an infinity of wffs breaking the "variable-sharing property" (VSP) (a logic L has the VSP if in all L-theorems of conditional form, antecedent and consequent share at least a propositional variable).

3. RM-semantics for PŁ4

In this section, PŁ4 is endowed with an RM-semantics (an RM-semantics without a set of designated points, in particular). Firstly, models and related notions are defined.

DEFINITION 3.1. A PŁ4RM-model (RM-model, for short) is a structure $(K, R, *, \vDash)$, where K is a set, R is a ternary relation on K and $*$ a unary operation on K subject to the following definitions and semantical postulates for all $a, b, c, d \in K$ with quantifiers ranging over K:

d1. $a \leq b =_{\text{df}} \exists x Rxab$

d1′. $a = b =_{\text{df}} a \leq b \ \& \ b \leq a$

d2. $R^2abcd =_{\text{df}} \exists x(Rabx \ \& \ Rxcd)$

P1. $a \leq a$

P2a. $(a \leq b \ \& \ Rbcd) \Rightarrow Racd$

P2b. $(a \leq b \ \& \ b \leq c) \Rightarrow a \leq c$

P2c. $(d \leq b \ \& \ Rabc) \Rightarrow Radc$

P2d. $(c \leq d \ \& \ Rabc) \Rightarrow Rabd$

P3. $R^2abcd \Rightarrow \exists x \exists y(Racx \ \& \ Rbcy \ \& \ Rxyd)$

P4. $Rabc \Rightarrow a \leq c$

P5. $Rabc \Rightarrow b \leq a$

P6. $a \leq b \Rightarrow b^* \leq a^*$

P7. $a = a^{**}$

Finally, $\vDash$ is a (valuation) relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i) $(a \leq b \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \to B$ iff for all $b, c \in K, (Rabc \ \& \ b \vDash A) \Rightarrow c \vDash B$

(iii) $a \vDash \neg A$ iff $a^* \nvDash A$

DEFINITION 3.2 (PŁ4RM-consequence, PŁ4RM-validity). For a non-empty set of wffs Γ and wff A, $\Gamma \vDash_M A$ (A is a consequence of Γ in the RM-model M) iff for all $a \in K$ in M, $a \vDash A$ whenever $a \vDash \Gamma$ ($a \vDash \Gamma$ iff $a \vDash B$ for all $B \in \Gamma$). Then, $\Gamma \vDash_{\text{RM}} A$ (A is a PŁ4RM-consequence — RM-consequence, for short — of Γ) iff $\Gamma \vDash_M A$ in every RM-model M.

In particular, if $\mathit{\Gamma} = \emptyset$, $\vDash_M A$ (A is true in M) iff $a \vDash A$ for all $a \in K$ in M. And $\vDash_{\text{PŁ4}} A$ (A is PŁ4RM-valid, RM-valid, for short) iff $\vDash_M A$ in every RM-model M.

In the sequel, we proceed to the proof of the soundness theorem. The two ensuing lemmas and proposition are useful.

Lemma 3.1 (Hereditary Lemma). *For any* RM-*model,* $a, b \in K$ *and any wff* A, $(a \leq b \ \& \ a \vDash A) \Rightarrow b \vDash A$.

Proof. Induction on the length of A. The conditional case is proved with P2a and the negation case is proved with P6. □

Lemma 3.2 (Entailment Lemma). *For any wffs* A, B, $\vDash_{\text{RM}} A \to B$ *iff* $(a \vDash A \Rightarrow a \vDash B$ *for all* $a \in K)$ *in all* RM-*models.*

Proof. (⇒) By P1. (⇐) By Lemma 3.1. □

Proposition 3.1. *The following semantical postulates are provable in any* RM-*model, for all* $a, b, c, d, e \in K$:

(P8) $Raaa$
(P9) $Rabc \Rightarrow b \leq c$
(P10) $Rabc \Rightarrow Rbac$
(P11) $(Rabc \ \& \ Ra^*de) \Rightarrow d \leq b^*$.

Proof. (P8) $Raaa$: By P1, and d1, (1) $Rxaa$. By 1 and P5, (2) $a \leq x$. Finally, by 1, 2 and P2a, (3) $Raaa$.

(P9) $Rabc \Rightarrow b \leq c$: Suppose (1) $Rabc$. By P5, (2) $b \leq a$. By P2a, 1 and 2. (3) $Rbbc$, whence by P4, (4) $b \leq c$ follows.

(P10) $Rabc \Rightarrow Rbac$: Suppose (1) $Rabc$. By P4, (2) $a \leq c$. By P8, (3) $Rccc$. By P2c, 2 and 3, (4) $Rcac$. By d2, 1 and 4, (5) R^2abac, whence, by P3, we have (6) $Raad$, (7) $Rbae$ and (8) $Rdec$ for some $d, e \in K$. By P9 and 8, (9) $e \leq c$. Finally, by P2d, 7 and 9, (10) $Rbac$, as desired.

(P11) $(Rabc \ \& \ Ra^*de) \Rightarrow d \leq b^*$: Suppose (1) $Rabc$ and (2) Ra^*de. By P5 and 1, (3) $b \leq a$. By P5 and 2, (4) $d \leq a^*$. By P6 and 3, (5) $a^* \leq b^*$. Finally, by P2b, 4 and 5, (6) $d \leq b^*$ follows. □

Next, the soundness theorem is proved.

Theorem 3.1 (Soundness of PŁ4). *For any set of wffs* $\mathit{\Gamma}$ *and any wff* A, *if* $\mathit{\Gamma} \vdash_{\text{PŁ4}} A$, *then* $\mathit{\Gamma} \vDash_{\text{RM}} A$.

Proof. If $A \in \mathit{\Gamma}$, the proof is trivial, and if A has been derived by MP, the proof is immediate by using P8. Concerning the RM-validity of the axioms, the proof of A4 and A5 is immediate by P7, and A1 and A2 are proved with P4 and P3, respectively [cf. 13, Chapter 4; 11, Proposition 6.5]. So, let us prove A3, A6, A7 and A8 (we lean upon Lemmas 3.1 and 3.2).

A3, $[(A \to B) \to A] \to A$, is RM-valid: Let M be an arbitrary RM-model where $a \in K$ and A, B be wffs such that (1) $a \vDash (A \to B) \to A$ but (2) $a \nvDash A$. By P8 ($Raaa$), 1 and 2, we have (3) $a \nvDash A \to B$, whence there are $b, c \in K$ such that (4) $Rabc$, (5) $b \vDash A$ and (6) $c \nvDash B$. By P5 and 4, (7) $b \leq a$ follows, whence by 5 we get (8) $a \vDash A$, contradicting 2.

A6, $\neg(A \to B) \to (\neg A \to C)$, is RM-valid: Let M be an arbitrary RM-model where $a \in K$ and A, B, C be wffs such that (1) $a \vDash \neg(A \to B)$ but (2) $a \nvDash \neg A \to C$. By 2, there are $b, c \in K$ such that (3) $Rabc$, (4) $b \vDash \neg A$ (i.e., $b^* \nvDash A$) and (5) $c \nvDash C$. On the other hand, by 1, we have (6) $a^* \nvDash A \to B$, i.e., (7) Ra^*de, (8) $d \vDash A$ and (9) $e \nvDash B$ for some $d, e \in K$. But, by P11, 3 and 7, (10) $d \leq b^*$ follows, whence by 8, we get (11) $b^* \vDash A$, contradicting 4.

A7, $\neg(A \to B) \to \neg B$, is RM-valid: Let M be an arbitrary RM-model where $a \in K$ and A, B be wffs such that (1) $a \vDash \neg(A \to B)$ but (2) $a \nvDash \neg B$ (i.e., $a^* \vDash B$). By 1, we have (3) $a^* \nvDash A \to B$, whence there are $b, c \in K$ such that (4) Ra^*bc, (5) $b \vDash A$ and (6) $c \nvDash B$. By P4 and 4, we get (7) $a^* \leq c$, whence by 2, we have (8) $c \vDash B$, contradicting 6.

A8, $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)$, is RM-valid: Let M be an arbitrary RM-model where $a \in K$ and A, B be wffs such that (1) $a \vDash \neg B$ (i.e., $a^* \nvDash B$) but (2) $a \nvDash [\neg A \to \neg(A \to B)] \to \neg(A \to B)$. By 2, there are $b, c \in K$ such that (3) $Rabc$, (4) $b \vDash \neg A \to \neg(A \to B)$ and (5) $c \nvDash \neg(A \to B)$. By P10 and 3 (6) $Rbac$ follows. Hence, by 4 and 5, we have (7) $a \nvDash \neg A$ (i.e., $a^* \vDash A$). On the other hand, by 5, we get (8) $c^* \vDash A \to B$; and by 3, P4 and P6, (9) $c^* \leq a^*$, whence by 8, we have (10) $a^* \vDash A \to B$. Finally, by P8 ($Ra^*a^*a^*$), 7 and 10, (11) $a^* \vDash B$ follows contradicting 1. □

4. Completeness of PŁ4

By using a canonical model construction, we prove the completeness of PŁ4 w.r.t. the general RM-semantics provided in the preceding section. In the first place, we define the notion of a theory and the classes of theories of interest in the present paper.

Definition 4.1. A PŁ4-theory (theory, for short) is a set of wffs containing all theorems of PŁ4 and closed under Modus Ponens (MP). That is, a is a theory iff (1) if $\vdash_{\text{PŁ4}} A$, then $A \in a$; and (2) $B \in a$ whenever $A \to B \in a$ and $A \in a$.

Definition 4.2 (Classes of PŁ4-theories). Let a be a theory. We set (1) a is prime iff whenever $(A \to B) \to B \in a$, then $A \in a$ or $B \in a$; (2) a is trivial if a contains all wffs; (3) a is a-consistent ('consistent in an absolute sense') iff a is not trivial; (4) a is w-inconsistent ('inconsistent in a weak sense') iff $\neg A \in a$, A being some PŁ4-theorem; (5) a is w-consistent ('consistent in a weak sense') iff a is not w-inconsistent (cf. [11] and references therein on the notion of w-consistency).

We prove a couple of useful propositions.

Proposition 4.1 (Closure under Adj and PŁ4-ent). *Let a be a theory. Then a is closed under Adjunction* (Adj) *and* PŁ4-*entailment* (PŁ4-ent). *That is,* (1) *if $A \in a$ and $B \in a$, then* $\neg[(\neg A \to \neg B) \to \neg B] \in a$; *and* (2) *if $\vdash_{\text{PŁ4}} A \to B$ and $A \in a$, then $B \in a$.*

Proof. Closure under PŁ4-ent: It is immediate since a contains all PŁ4-theorems and it is closed under MP.

Closure under Adj: Immediate by T2, $A \to [B \to \neg[(\neg A \to \neg B) \to \neg B]]$ and closure under PŁ4-ent and MP. □

Proposition 4.2. *For any theory a, a is w-consistent iff a is a-consistent.*

Proof. Immediate by Efq$_2$, $\vdash_{\text{PŁ4}} A \Rightarrow \vdash_{\text{PŁ4}} \neg A \to B$. □

Next, the canonical model is defined.

Definition 4.3 (The canonical PŁ4RM-model). Let K^T be the set of all theories and R^T be defined on K^T as follows: for any $a, b, c \in K^T$, $R^T abc$ iff for any wffs A, B, $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in c$. Next, let K^C be the set of all a-consistent prime theories. On the other hand, let R^C be the restriction of R^T to K^C and $*^C$ be defined on K^C as follows: for each $a \in K^C$, $a^{*^C} = \{A \mid \neg A \notin a\}$. Finally, the relation $\vDash^C$ is defined as follows for any wff A and $a \in K^C$: $a \vDash^C A$ iff $A \in a$. Then, the canonical PŁ4RM-model (canonical RM-model, for short) is the structure $(K^C, R^C, *^C, \vDash^C)$.

We need to show that the canonical model is indeed a model. And in order to do this, the following facts have to be proven: (1) the set K^C is not empty; (2) $*^C$ is an operation on K^C; (3) the semantical postulates P1-P7 are canonically valid; (4) the conditions (i)-(iii) in Definition 3.1 hold canonically. Well then, in the sequel, we proceed to prove these facts. We begin by proving the primeness lemma.

Lemma 4.1 (Extension to prime theories). *Let a be a theory and A a wff such that $A \notin a$. Then, there is a prime theory x such that $a \subseteq x$ and $A \notin x$.*

Proof. Cf. [13, Chapter 4], where it is shown how to proceed in the case of any logic including Routley and Meyer's basic positive logic B_+. In particular, a proof in the case of PŁ4 is provided in Lemma 3.9 in [8]. □

Corollary 4.1 (Non-emptyness of K^C). *The set K^C is not empty.*

Proof. Immediate by Lemma 4.1, since $PŁ4_{TH}$ is an a-consistent theory ($PŁ4_{TH}$ is the set of all theorems of PŁ4). □

Lemma 4.2 ($*^C$ is an operation on K^C). *Let a be an a-consistent prime theory. Then, a^{*^C} is an a-consistent prime theory as well.*

Proof. (In this proof and in the rest of the section the superscript C is generally dropped from above $*$, R and $\vDash$ when there is no risk of confusion). (a) a^{*^C} is closed under MP: Suppose (1) $A \to B \in a^*$ (i.e., $\neg(A \to B) \notin a$), (2) $A \in a^*$ (i.e., $\neg A \notin a$) but (3) $B \notin a^*$ (i.e., $\neg B \in a$). By A8, $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)]$ and 3, we have (4) $[\neg A \to \neg(A \to B)] \to \neg(A \to B) \in a$, whence, by the primeness of a, (5) $\neg(A \to B) \in a$ or $\neg A \in a$ follows. But 1 and 2 contradict 5. (b) a^{*^C} contains all PŁ4-theorems: Suppose $A \notin a^*$, A being a PŁ4-theorem. Then, we have $\neg A \in a$, contradicting the w-consistency of a (cf. Proposition 4.2). (c) a^{*^C} is a-consistent: Suppose $\neg A \in a^*$, A being a PŁ4-theorem. Then we have $\neg\neg A \notin a$, whence by A4, $A \notin a$ follows, contradicting the fact that a contains all PŁ4-theorems. (d) a^{*^C} is prime: Immediate by T3, $\neg B \to [\neg A \to \neg[(A \to B) \to B]]$. □

In order to show that the semantical postulates and the clauses hold canonically, we need to prove some preliminary facts. We begin by defining an alternative concept of a PŁ4-theory equivalent to the one in Definition 4.1. This alternative but equivalent notion of a PŁ4-theory is convenient for using some known results on RM-semantics.

Definition 4.4 (PŁ4-theory 2). A PŁ4-theory 2 (theory 2, for short) is a set of wffs containing all PŁ4-theorems and closed under Adj and PŁ4-ent (cf. Proposition 4.1).

Proposition 4.3 (Closure under MP). *Let a be a theory 2. Then, a is closed under MP.*

Proof. Suppose (1) $A \to B \in a$ and (2) $A \in a$. As a is closed under Adj, (3) $\neg[[\neg(A \to B) \to \neg A] \to \neg A] \in a$ (cf. Proposition 4.1). Then, $B \in a$ follows by T4, $\neg[[\neg(A \to B) \to \neg A] \to \neg A] \to B$. □

The next proposition states that the two notions of a PŁ4-theory are equivalent. Indeed, immediate by Definitions 4.1, 4.4 and Propositions 4.1, 4.3. we obtain:

Proposition 4.4. *For any set a of wffs, a is a theory iff a is a theory 2.*

In the sequel, we lean upon some results in [11], where some facts about logics including the basic positive logic $\mathrm{B_{K+}}$ are proven ($\mathrm{B_{K+}}$ is defined in the appendix).

Let L be a logic including $\mathrm{B_{K+}}$, an L-theory be a non-empty set of wffs closed under Adj and L-ent (cf. Definition 4.4), and K^T and R^T be defined similarly as in Definition 4.3. Moreover, let K^P be the set of all prime L-theories containing all L-theorems and R^P be the restriction of R^T to K^P. We have (cf. [11, §3.2]; a is prime if $A \in a$ or $B \in a$ whenever $A \vee B \in a$):

Proposition 4.5. *The following are some facts about L:*

1. *Let $a, b \in K^P$, $c \in K^T$ and $R^T abc$. Then, there is some $x \in K^P$ such that $c \subseteq x$ and $R^P abx$.*
2. *Let $a, b \in K^T$, $c \in K^P$ and $R^T abc$. Then, there are $x, y \in K^P$ such that $a \subseteq x$, $b \subseteq y$ and $R^P xyc$.*
3. *Let $a, b \in K^P$. Then, $a \leq^P b$ iff $a \subseteq b$ ($a \leq^P b =_{\mathrm{df}} \exists x \in K^P\ R^P xab$).*

On the other hand, we prove:

Proposition 4.6 (a-consistency in $R^T abc$). *Let a, b be* PŁ4*-theories, c an a-consistent prime* PŁ4*-theory and $R^T abc$. Then a and b are a-consistent as well.*

Proof. (Cf. Proposition 4.2.) (a) a is a-consistent: Suppose (1) $R^T abc$ but (2) $\neg A \in a$, A being a PŁ4-theorem. In addition, let (3) C be a PŁ4-theorem as well and (4) $B \in b$. By $\mathrm{Efq_2}$, we have (5) $\vdash_{\text{PŁ4}} \neg A \to$

$(B \to \neg C)$, whence by 2, we have (6) $B \to \neg C \in a$ and hence by 1 and 4, we get (7) $\neg C \in c$, contradicting the a-consistency of c.

(b) b is a-consistent: Suppose (1) $R^T abc$ but (2) $\neg A \in b$, A being a PŁ4-theorem. By T1, we have (3) $\neg A \to \neg A \in a$, whence by 1 and 2, we get (4) $\neg A \in c$, contradicting the a-consistency of c. □

Given Propositions 4.5 and 4.6, we have the following corollary on PŁ4-theories (cf. Definition 4.3).

COROLLARY 4.2. *The following are some facts about the canonical* RM-*model*:

1. *If* $a, b \in K^C$, $c \in K^T$ *and* $R^T abc$, *then there is an* $x \in K^C$ *such that* $c \subseteq x$ *and* $R^C abx$.
2. *If* $a, b \in K^T$, $c \in K^C$ *and* $R^T abc$, *then there are* $x, y \in K^C$ *such that* $a \subseteq x$, $b \subseteq y$ *and* $R^C xyc$.
3. *For any* $a, b \in K^C$, $a \leq^C b$ *iff* $a \subseteq b$ (where $a \leq^C b =_{\mathrm{df}} \exists x \in K^C$ $R^C xab$).

Now we can prove the canonical validity of the semantical postulates and the clauses in Definition 3.1.

LEMMA 4.3. *The semantical postulates* P1, P2a, P2b, P2c, P2d, P3, P4, P5, P6 *and* P7 *are satisfied by the canonical* RM-*model.*

PROOF. By using Corollary 4.2, the proof of P1, P2a, P2b, P2c, P2d and P6 is trivial, while that of P7 is immediate (by A4 and A5). Moreover, the same corollary can be applied to greatly simplify the proofs of P3, P4 and P5. In particular, P3, P4 and P5 are proved in [11, Proposition 6.5], if we bear in mind that a PŁ4-theory can be understood as stated in Definition 4.4 (Proposition 6.5 in [11] can be proved for any logic L including $\mathrm{B_{K+}}$, provided L has A2 (resp. A1, A3) in order to show the canonical validity of P3 (resp. P4, P5). L-theories are understood as non-empty sets of wffs closed under Adj and L-ent). □

LEMMA 4.4. *The conditions* (*clauses*) (i)–(iii) *in Definition 3.1 are satisfied by the canonical* RM-*model.*

PROOF. Clause (i) is immediate by Corollary 4.2(3) and clause (iii) is trivial by Definition 4.3. Then, clause (ii) is proved in [11, Lemma 3.20] using Corollary 4.2(1,2) and the fact that a PŁ4-theory can be understood as a set of wffs closed under Adj and PŁ4-ent, and containing all PŁ4-theorems (cf. Definition 4.4). □

Immediate by Corollary 4.1 and Lemmas 4.2, 4.3 and 4.4 we have:

Corollary 4.3. *The canonical* RM-*model is indeed an* RM-*model.*

Finally, the completeness theorem is proved. We lean on the standard notion of 'the set of consequences of a set of wffs'.

Definition 4.5 (The set CnΓ[PŁ4]). The set of consequences in PŁ4 of a set of wffs Γ (in symbols, CnΓ[PŁ4]) is defined as follows. CnΓ[PŁ4] $=$ $\{A \mid \Gamma \vdash_{\text{PŁ4}} A\}$.

Remark 4.1. For any set of wffs Γ, CnΓ[PŁ4] is a PŁ4-theory.

Theorem 4.1 (Completeness of PŁ4). *For any set of wffs Γ and wff A, if $\Gamma \vDash_{\text{RM}} A$, then $\Gamma \vdash_{\text{PŁ4}} A$.*

Proof. Suppose $\Gamma \nvdash_{\text{PŁ4}} A$. Then $A \notin$ CnΓ[PŁ4]. By Lemma 4.1, there is a prime theory $\mathcal{T}$ such that $\Gamma \subseteq$ CnΓ[PŁ4] $\subseteq \mathcal{T}$ and $A \notin \mathcal{T}$. Given that $\mathcal{T} \in K^C$ and that the canonical RM-model is an RM-model, we have $\Gamma \nvDash^C A$, since $\mathcal{T} \vDash^C \Gamma$ but $\mathcal{T} \nvDash^C A$. Then, $\Gamma \nvDash_{\text{RM}} A$ follows by Definition 3.2.

If Γ is empty, the proof is similar, since PŁ4$_{\text{TH}}$ is an a-consistent theory (PŁ4$_{\text{TH}}$ is the set of all PŁ4-theorems). □

5. 2 set-up RM-semantics for PŁ4

In [2], 2 set-up RM-semantics (2RM-semantics, for short) is introduced and the logics BN4, RM3 and Łukasiewicz's 3-valued logic Ł3 are interpreted with this type of semantics. In [12], the logic E4 is also given a 2RM-semantics. The aim of this section is to add PŁ4 to this limited group of logics by endowing it with a 2RM-semantics, greatly simplifying the general RM-semantics. We begin by defining the concept of a model and related notions.

Definition 5.1 (PŁ42RM-models). Let $*$ be an involutive operation defined on the set K, that is, for any $a \in K$, $a = a^{**}$, and let K be the two-element set $\{0, 0^*\}$. A PŁ42RM-model (2RM-model, for short), i.e., a 2 set-up Routley-Meyer PŁ4-model, is a structure $(K, R, *, \vDash)$, where:

(I) R is a ternary relation on K subject to the following definition and semantical postulates for all $a, b, c \in K$: (d1) $a \leq b =_{\text{df}} \exists x \in K Rxab$; (I1) $(a \leq b\ \&\ Rbcd) \Rightarrow Racd$; (I2) $Raaa$. (I3) $Rabc \Rightarrow Rbac$;

(II) $\vDash$ is a (valuation) relation such that conditions (i), (ii) and (iii) are as in Definition 3.1.

Finally, the notions of PŁ42RM-consequence (2RM-consequence, for short) and PŁ42RM-validity (2RM-validity, for short) are defined in a similar way to which RM-consequence and RM-validity are defined in Definition 3.2.

Then, P1, P2b, P2c, P3, P4, P5 and P6 (cf. Definition 3.1) are easily provable, while P2d, although not necessary in the completeness proof, can safely be added since it is trivially proved when canonically interpreted. Next, the Hereditary and Entailment lemmas are proved similarly as in RM-semantics (§2). On the other hand, we have the following useful proposition.

PROPOSITION 5.1. *For any* 2RM-*model, clause* (ii) *can be simplified to the following clause* (ii′): *For any* $a \in K$ *and wffs* A, B, $a \vDash A \to B$ *iff* $a \nvDash A$ *or* $a \vDash B$.

PROOF. Let M be an arbitrary 2RM-model where $a \in K$ ($\Rightarrow$) Suppose (1) $a \vDash A \to B$ and (2) $a \vDash A$. By I2, we have (3) $Raaa$. By 1, 2 and 3, we get (4) $a \vDash B$. ($\Leftarrow$) Suppose (1) $a \nvDash A$ or $a \vDash B$ and for any $b, c \in K$, (2) $Rabc$ and (3) $b \vDash A$. We need to prove $c \vDash B$. By P5, P4 and 2 we have (4) $b \leq a$ and (5) $a \leq c$, respectively. By 3 and 4, we get (6) $a \vDash A$, whence (7) $a \vDash B$ follows by 1. Finally, we have, by 5 and 7, (8) $c \vDash B$, as required. □

Now, we can prove the soundness theorem.

THEOREM 5.1 (Soundness of PŁ4 w.r.t. 2RM-semantics). *For any set of wffs* Γ *and wff* A, *if* $\Gamma \vdash_{\text{PŁ4}} A$, *then* $\Gamma \vDash_{\text{2RM}} A$.

PROOF. Similar to (but simpler than) that of Theorem 3.1. Let us, for example, prove the 2RM-validity of A8 (we use Proposition 5.1).

A8, $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)]$, is 2RM-valid. Let M be an arbitrary 2RM-model where $a \in K$ and A, B wffs such that (1) $a \vDash \neg B$ (i.e., $a^* \nvDash B$) but (2) $a \nvDash [\neg A \to \neg(A \to B)] \to \neg(A \to B)$. By 2, we have (3) $a \vDash \neg A \to \neg(A \to B)$ and (4) $a \nvDash \neg(A \to B)$. By 3 and 4, we get (5) $a \nvDash \neg A$ (i.e., $a^* \vDash A$); and by 4, we obtain (6) $a^* \vDash A \to B$, whence by 1, we have (7) $a^* \nvDash A$, contradicting 5. □

Turning to completeness, we suppose that Γ is a set of wffs and A is wff such that $\Gamma \nvdash_{\text{PŁ4}} A$ and then we prove $\Gamma \nvDash_{\text{2RM}} A$.

Suppose then $\Gamma \nvdash_{\text{PŁ4}} A$. Proceeding similarly as in Theorem 4.1, it is shown that there is a prime theory $\mathcal{T}$ such that $\Gamma \subseteq \mathcal{T}$ and $A \notin \mathcal{T}$.

The canonical PŁ42RM-model (2RM-model, for short) is defined as the structure $(K^C, R^C, *^C, \vDash^C)$ where $K^C = \{\mathcal{T}, \mathcal{T}^{*^C}\}$, $\mathcal{T}$ being the theory just built up and R^C, $*^C$ and $\vDash^C$ being defined similarly as in the canonical RM-model (Definition 4.3).

Then, in order to show that the canonical 2RM-model is a 2RM-model, we need to show: (1) Postulates I1, I2 and I3 hold canonically. (2) $*^C$ is an involutive operation on K^C. (3) Conditions (i), (ii) and (iii) in Definition 5.1 hold in the canonical 2RM-model. Now, (1) is proved similarly as in Lemma 4.3 by using Corollary 4.2(3); and (2) follows by Lemma 4.2 and A4, A5. Concerning (3), (i) is trivial and (iii) is directly derivable from the definition of the canonical 2RM-model. Finally, (ii) is proved as follows (cf. Proposition 5.1). (a) (⇒) Suppose that A and B are wffs such that $A \to B \in \mathcal{T}$ and $A \in \mathcal{T}$. Then, $B \in \mathcal{T}$ is immediate by closure of $\mathcal{T}$ under MP. (a) (⇐) Suppose that A and B are wffs such that (1) $A \to B \notin \mathcal{T}$. We have to prove $A \in \mathcal{T}$ and $B \notin \mathcal{T}$. For reductio, assume (2) $A \notin \mathcal{T}$ or (3) $B \in \mathcal{T}$. By A3, $[(A \to B) \to A] \to A$, and the primeness of $\mathcal{T}$, we have (4) either $A \to B \in \mathcal{T}$ or $A \in \mathcal{T}$. But 1 and 2 contradict 4. On the other hand, given 3 and A1, $B \to (A \to B)$, we get (5) $A \to B \in \mathcal{T}$, contradicting 1. Thus, $A \in \mathcal{T}$ and $B \notin \mathcal{T}$, as was to be proved. (b) (⇒) Suppose that A and B are wffs such that (1) $A \to B \in \mathcal{T}^*$ (i.e., $\neg(A \to B) \notin \mathcal{T}$) and (2) $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$) and, for reductio, (3) $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$). By A8, $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)]$ and 3, we have (4) $[\neg A \to \neg(A \to B)] \to \neg(A \to B) \in \mathcal{T}$, whence by the primeness of $\mathcal{T}$, we get (5) either $\neg(A \to B) \in a$ or $\neg A \in a$. But 1 and 2 contradict 5. (b) (⇐) Suppose that A and B are wffs such that (1) $A \to B \notin \mathcal{T}^*$ (i.e., $\neg(A \to B) \in \mathcal{T}$). We have to prove $A \in \mathcal{T}^*$ (i.e., $\neg A \notin \mathcal{T}$) and $B \notin \mathcal{T}^*$ (i.e., $\neg B \in \mathcal{T}$). By A7, $\neg(A \to B) \to \neg B$ and 1, we have (2) $\neg B \in \mathcal{T}$. On the other hand, for reductio, suppose (3) $A \notin \mathcal{T}^*$ (i.e., $\neg A \in \mathcal{T}$). By A6, $\neg(A \to B) \to (\neg A \to C)$, and 3, we get $C \in \mathcal{T}$ for any wff C, contradicting the a-consistency of $\mathcal{T}$. Thus, $A \in \mathcal{T}^*$ and $B \notin \mathcal{T}^*$, as was to be proved.

With the canonical 2RM-model having been shown a 2RM-model, the completeness of PŁ4 w.r.t. the 2RM-semantics is proved similarly as in Theorem 4.1.

6. Binary Routley semantics and 2 set-up binary Routley semantics for PŁ4

In this section, PŁ4 is given both a binary Routley semantics (bR-semantics) and a 2 set-up binary Routley semantics (2bR-semantics). Firstly, the bR-semantics is developed.

DEFINITION 6.1 (PŁ4bR-models). A PŁ4bR-model (bR-model for short) is a structure $(K, R, *, \vDash)$ where K is a non-empty set, R is a binary relation on K and $*$ a unary operation on K subject to the following postulates for all $a, b \in K$:

P1. Raa

P2. $(Rab \ \& \ Rbc) \Rightarrow Rac$

P3. $Rab \Rightarrow Rba$

P4. $Rab \Rightarrow Rb^*a^*$

P5. Raa^{**}

P6. $Ra^{**}a$

Finally, $\vDash$ is a (valuation) relation from K to the set of all wffs such that the following conditions (clauses) are satisfied for every propositional variable p, wffs A, B and $a \in K$:

(i) $(Rab \ \& \ a \vDash p) \Rightarrow b \vDash p$

(ii) $a \vDash A \to B$ iff for all $b \in K, (Rab \ \& \ b \vDash A) \Rightarrow b \vDash B$

(iii) $a \vDash \neg A$ iff $a^* \nvDash A$

Once PŁ4bR-consequence (bR-consequence) and PŁ4bR-validity (bR-validity) are defined similarly as PŁ4RM-consequence and PŁ4RM-validity in Definition 3.2, the soundness proof mirrors that in Section 3. We have:

LEMMA 6.1 (Hereditary condition). *For any* bR-*model,* $a, b \in K$ *and any wff* A, $(Rab \ \& \ a \vDash A) \Rightarrow b \vDash A$.

PROOF. Induction on the structure of A. The conditional case is proved by P2 and the negation case, by P4. □

Trivial, by P1, we obtain:

LEMMA 6.2 (Entailment). *For any wffs* A, B, $\vDash_{\text{bR}} A \to B$ *iff* $(a \vDash A \Rightarrow a \vDash B$ *for all* $a \in K)$ *in all models.*

PROPOSITION 6.1. *The following semantical postulate* P7 *is provable in any* bR-*model, for all* $a, b, c \in K$: (P7) $(Ra^*c \ \& \ Rab) \Rightarrow Rc^*b$.

PROOF. Suppose (1) Ra^*c and (2) Rab. By P4 and 1, we have (3) Rc^*a^{**}. By P2, P6 and 3, (4) Rc^*a. Finally, we have (5) Rc^*b (by P2, 2 and 4), as desired. □

Next, the soundness theorem is proved.

THEOREM 6.1 (Soundness of PŁ4). *For any set of wffs* Γ *and wff* A, *if* $\Gamma \vdash_{\text{PŁ4}} A$, *then* $\Gamma \vDash_{\text{bR}} A$.

PROOF. If $A \in \Gamma$, the proof is trivial, and if A has been derived by MP, the proof is immediate by using P1. Regarding the bR-validity of the axioms, A1, A2, A3, A4 and A5 are immediate: A1, by Lemma 6.2; A2, by P1 and P2; A3, by P3 and Lemma 6.1; and A4 (resp., A5) by P5 (resp., P6). So, let us prove A6, A7 and A8 (we use Lemmas 6.1 and 6.2).

A6, $\neg(A \to B) \to (\neg A \to C)$, is bR-valid: Let M be an arbitrary bR-model where $a \in K$ and let A, B, C be wffs such that (1) $a \vDash \neg(A \to B)$ but (2) $a \nvDash \neg A \to C$. By 2, we have for some $b \in K$ (3) Rab, (4) $b \vDash \neg A$ (i.e., $b^* \nvDash A$) and (5) $b \nvDash C$. On the other hand, by 1, we have (6) $a^* \nvDash A \to B$, i.e., for some $c \in K$ (7) Ra^*c, (8) $c \vDash A$ and (9) $c \nvDash B$. But, by P7, 3 and 7, (10) Rcb^* follows, whence by 8, we get (11) $b^* \vDash A$, contradicting 4.

A7, $\neg(A \to B) \to \neg B$, is bR-valid: Let M be an arbitrary bR-model where $a \in K$ and A, B are wffs such that (1) $a \vDash \neg(A \to B)$ but (2) $a \nvDash \neg B$ (i.e., $a^* \vDash B$). By 1, we have (3) $a^* \nvDash A \to B$, whence there is $b \in K$ such that (4) Ra^*b, (5) $b \vDash A$ and (6) $b \nvDash B$. But by 2 and 4, we have (7) $b \vDash B$, contradicting 6.

A8, $\neg B \to [[\neg A \to \neg(A \to B)] \to \neg(A \to B)]$, is bR-valid: Let M be an arbitrary bR-model where $a \in K$ and A, B are wffs such that (1) $a \vDash \neg B$ (i.e., $a^* \nvDash B$) but (2) $a \nvDash [\neg A \to \neg(A \to B)] \to \neg(A \to B)$. By (2), there is $b \in K$ such that (3) Rab, (4) $b \vDash \neg A \to \neg(A \to B)$ and (5) $b \nvDash \neg(A \to B)$ (i.e., $b^* \vDash A \to B$). By P4 and 3, we have (6) Rb^*a^*; and by 1, 5 and 6, (7) $a^* \nvDash A$ (i.e., $a \vDash \neg A$). Moreover, by P3 and 3, we get (8) Rba. Then, by 4, 7 and 8, (9) $a \vDash \neg(A \to B)$ follows, whence by 3, we have (10) $b \vDash \neg(A \to B)$, contradicting 5. □

Turning to completeness, the proof is based on a canonical model construction, just as done in Section 4. The canonical PŁ4bR-model (canonical bR-model, for short) is a structure $(K^C, R^C, *^C, \vDash^C)$, where $K^C, R^C, *^C, \vDash^C$ are defined as in Definition 4.3, except for the relation R^T that now reads as follows: for any $a, b \in K^T$ and wffs A, B, $R^T ab$ iff $(A \to B \in a \ \ \& \ \ A \in b) \Rightarrow B \in b$. On the other hand, it is clear the we have at our disposal the facts proved in Section 4: closure of theories under Adj and PŁ4-entailment (Proposition 4.1), coextensiveness of a-consistent and w-consistent theories (Proposition 4.2), extension of a-consistent theories to prime theories (Lemma 4.1), non-emptiness of K^C (Corollary 4.1), $*^C$ is an operation on K^C (Lemma 4.2). Thus, it remains to prove: (1) the semantical postulates P1-P6 are canonically valid, and (2) conditions (i)-(iii) in Definition 6.1 hold canonically.

In order to prove that the semantical postulates hold canonically, the following alternative way of interpreting the canonical relation R^C proves to be useful.

PROPOSITION 6.2. *For all $a, b \in K^C$, $R^C ab$ iff $a \subseteq b$.*

PROOF. ($\Rightarrow$) Suppose (1) $R^C ab$, (2) $A \in a$ and let (3) $B \in b$. We prove $A \in b$. By A1, (4) $A \to (B \to A)$ is a theorem, whence by 2, we get (5) $B \to A \in a$. Finally, by 1, 3 and 5, (6) $A \in b$ follows.

($\Leftarrow$) Suppose (1) $a \subseteq b$, (2) $A \to B \in a$ and (3) $A \in b$. We prove $B \in b$. By 1 and 2, we have (4) $A \to B \in b$, and by closure under Adj, 3 and 4, (5) $\neg[[\neg(A \to B) \to \neg A] \to \neg A] \in b$, whence by T4, $\neg[[\neg(A \to B) \to \neg A] \to \neg A] \to B$, (6) $B \in b$ follows, as desired. □

LEMMA 6.3. *The semantical postulates* P1–P6 *are satisfied by the canonical* bR-*model.*

PROOF. We use Proposition 6.2. P1 and P2 are trivial by Proposition 6.2; and P5 and P6 are immediate by A4 and A5, respectively. So, let us prove P3 and P4.

P3, $R^C ab \Rightarrow R^C ba$, holds in the canonical bR-model: Let $a, b \in K^C$ and suppose (1) $R^C ab$ and (2) $A \in b$ but (3) $A \notin a$. By A3, we have (4) $[(A \to B) \to A] \to A \in a$ for an arbitrary wff B; and by the primeness of a, (5) either $A \to B \in a$ or $A \in a$. Thus, (6) $A \to B \in a$ follows by 3. Then, by 1, 2 and 6, we have (7) $B \in b$, contradicting the a-consistency of b.

P4, $R^C ab \Rightarrow R^C b^* a^*$, holds in the canonical bR-model: Let $a, b \in K^C$ and suppose (1) $R^C ab$ and (2) $A \in b^*$ (i.e., $\neg A \notin b$). Then, (3) $\neg A \notin a$ by 1 and 2, whence (4) $A \in a^*$, as required. □

Lemma 6.4. *The conditions* (*clauses*) (i)–(iii) *in Definition 6.1 are satisfied by the canonical* bR-*model.*

Proof. Clause (i) is immediate by Proposition 6.2, and clause (iii) and clause (ii) (from left to right) are also immediate, now by the definition of the canonical bR-model. So let us prove clause (ii) from right to left. Let $a \in K^C$ and A, B be wffs such that (1) $a \nvDash^C A \to B$ (i.e., $A \to B \notin a$). We prove that there is some $b \in K^C$ such that $R^C ab$, $b \vDash^C A$ (i.e., $A \in b$) and $b \nvDash^C B$ (i.e., $B \notin b$). So, consider the set $x = \{C \mid A \to C \in a\}$. By using A1, it is easy to show that x is a theory (i.e., x is closed under MP and contains all PŁ4-theorems) such that $R^T ax$ and $B \notin x$. Then, by the primeness lemma, x is extended to a prime theory b such that $R^C ab$, $A \in b$ and $B \notin b$, as was required. □

With the canonical bR-model proven to be a bR-model, the completeness proof proceeds similary as the completeness of PŁ4 w.r.t. RM-semantics in Section 4.

In the sequel, we introduce the notion of a 2 set-up Routley semantics (2bR-semantics) and give PŁ4 this type of semantics. 2RM-semantics and 2bR-semantics are essentially distinguished by the relation R defined on the set K of all points, which is ternary in the former and binary in the latter. Thus, PŁ42bR-models (2bR-models, for short) are defined similarly as 2RM-models in Definition 5.1, except that now R is defined as follows: R is a binary relation on K such that, for all $a, b \in K$, (I1) $Raaa$; (I2) $Rab \Rightarrow Rba$. The notions of PŁ42bR-consequence (2bR-consequence, for short) and PŁ42bR-validity (2bR-validity, for short) are defined similarly as in Definition 3.2.

Proceeding to the soundness and completeness proofs, we firstly note that P2 and P4 (cf. Definition 6.1) are easily proved and that the Hereditary and Entailment Lemmas are proved similarly as in bR-semantics; then that the simplification of clause (ii) to (ii′) (cf. Proposition 5.1) is easily adapted to 2bR-models from 2RM-models (cf. Proposition 5.1) and the soundness theorem can be proved similarly as in the case of 2RM-models (cf. Theorem 5.1). As regards completeness, we only need to prove that I1 and I2 hold canonically, since the clauses (i) and (ii′) are proved similarly as in 2RM-semantics (Section 5). But the canonical validity of the postulates is proved following the pattern set up in the case of 2RM-semantics by adjusting it now to the proof given for the general case of 2bR-semantics (Section 6).

7. Concluding remarks

In the present paper, PŁ4 is given both a Routley-Meyer ternary semantics and a binary Routley semantics of the kind established in [10], and also a 2 set-up ternary Routley-Meyer semantics and a 2 set-up binary Routley semantics. The latter kind of semantics is introduced in the present paper, PŁ4 being the first logic endowed with this type of semantics, to the best of our knowledge. It has to be noted that the relational semantics PŁ4 has been interpreted with in the present paper are simpler than the ones preceding them in the literature (cf. the introduction to the paper), given that PŁ4 is a strong logic. In this way, we hope to have shed new light on a logic already interpretable from more than one viewpoint, as remarked in the introduction to the paper.

Regarding future work on the topic, we note two suggestions, one on the logic PŁ4, the other one on 2 set-up binary Routley semantics. (1) The expansion of PŁ4 with necessity and possibility connectives defined by a binary accessibility relation introduced in 2bR-models, instead of defining said connectives with $\to$ and $\neg$ as shown in the appendix (similar to a corresponding expansion of the logic E4 as carried on in [12]). (2) There are essentially two ways of extending the relation R characteristic of 2bR-models: we can require $R00^*$ or else $R0^*0$ (addition of both conditions would cause the collapse into classical propositional logic). It would be interesting to investigate which logics are characterized by each one of these extensions of 2bR-semantics.

A. Appendix

The conjunction ($\wedge$), disjunction ($\vee$), necessity (L) and possibility (M) connectives given by the following tables:

$\wedge$	0	1	2	3
0	0	0	0	0
1	0	1	0	1
2	0	0	2	2
3	0	1	2	3

$\vee$	0	1	2	3
0	0	1	2	3
1	1	1	3	3
2	2	3	2	3
3	3	3	3	3

	L
0	0
1	0
2	0
3	3

	M
0	0
1	3
2	3
3	3

are definable in MPŁ4 by putting, for any wffs A, B: $A \vee B =_{\text{df}} (A \to B) \to B$; $A \wedge B =_{\text{df}} \neg(\neg A \vee \neg B)$; $LA =_{\text{df}} \neg(A \to \neg A)$; $MA =_{\text{df}} \neg L \neg A$.

Next, we list some theorems and rules of PŁ4. Firstly, notice that any theorem of negationless classical propositional logic is a theorem of

PŁ4, since the following wffs are provable in PŁ4: (t1) $A \to (A \vee B)$; (t2) $B \to (A \vee B)$; (t3) $(A \to C) \to [(B \to C) \to [(A \vee B) \to C)]$; (t4) $(A \wedge B) \to A$; (t5) $(A \wedge B) \to B$; (t6) $A \to [B \to (A \wedge B)]$. But A1–A3 (cf. §2) and t1–t6 axiomatize (together with MP) the negationless fragment of classical propositional logic. In addition, the following are also theorems and rules of PŁ4:

Con 1. $\vdash_{\text{PŁ4}} A \to B \Rightarrow\ \vdash_{\text{PŁ4}} \neg B \to \neg A$

Con 2. $\vdash_{\text{PŁ4}} A \to \neg B \Rightarrow\ \vdash_{\text{PŁ4}} B \to \neg A$

Con 3. $\vdash_{\text{PŁ4}} \neg A \to B \Rightarrow\ \vdash_{\text{PŁ4}} \neg B \to A$

Con 4. $\vdash_{\text{PŁ4}} \neg A \to \neg B \Rightarrow\ \vdash_{\text{PŁ4}} B \to A$

Efq_1. $\vdash_{\text{PŁ4}} \neg A \Rightarrow\ \vdash_{\text{PŁ4}} A \to B$

Efq_2. $\vdash_{\text{PŁ4}} A \Rightarrow\ \vdash_{\text{PŁ4}} \neg A \to B$

t7. $\neg(A \vee B) \leftrightarrow (\neg A \wedge \neg B)$

t8. $\neg(A \wedge B) \leftrightarrow (\neg A \vee \neg B)$

t9. $(A \vee B) \leftrightarrow \neg(\neg A \wedge \neg B)$

t10. $(A \wedge B) \leftrightarrow \neg(\neg A \vee \neg B)$

t11. $LA \leftrightarrow \neg M \neg A$

t12. $MA \leftrightarrow \neg L \neg A$

t13. $LA \to A$

t14. $A \to MA$

t15. $LA \to LLA$

t16. $MA \to LMA$

t17. $MLA \to LA$

t18. $L(A \to B) \to (LA \to LB)$

t19. $L(A \wedge B) \leftrightarrow (LA \wedge LB)$

t20. $M(A \vee B) \leftrightarrow (MA \vee MB)$

t21. $M(A \to B) \leftrightarrow (LA \to MB)$

t22. $(MA \to LB) \to L(A \to B)$

t23. $(MA \to MB) \to M(A \to B)$

t24. $(LA \vee LB) \to L(A \vee B)$

t25. $(MA \wedge MB) \to M(A \wedge B)$

t26. $L(A \vee B) \to (LA \vee MB)$

t27. $(MA \wedge LB) \to M(A \wedge B)$

t28. $A \vee \neg LA$

t29. $(LA \wedge \neg A) \to B$

t30. $A \to (\neg A \vee LA)$

Nec. $\vdash_{\text{PŁ4}} A \Rightarrow\ \vdash_{\text{PŁ4}} LA$

RT. $\vdash_{\text{PŁ4}} A \leftrightarrow B \Rightarrow\ \vdash_{\text{PŁ4}} C[A] \leftrightarrow C[A/B]$

DT. $\Gamma, A \vdash_{\text{PŁ4}} B \Rightarrow \Gamma \vdash_{\text{PŁ4}} A \to B$

(The biconditional ($\leftrightarrow$) is defined in the customary way: $A \leftrightarrow B =_{\text{df}} (A \to B) \wedge (B \to A)$. Con abbreviates Contraposition. Efq abbreviates 'E falso quodlibet' — Any proposition is implied by a false proposition. Nec abbreviates 'Necessitation' rule. RT abbreviates 'Replacement theorem': $C[A]$ is a wff where A appears; $C[A/B]$ is the result of changing one or more occurrences of A in $C[A]$ for corresponding occurrences of B. Finally, DT means 'Deduction Theorem'.)

Definition A.1 (The logic B_+). Routley and Meyer's basic positive logic B_+ can be axiomatized as follows [cf. 13].

Axioms:

A1. $A \to A$

A2. $(A \wedge B) \to A$ / $(A \wedge B) \to B$

A3. $[(A \to B) \wedge (A \to C)] \to [A \to (B \wedge C)]$

A4. $A \to (A \vee B)$ / $B \to (A \vee B)$

A5. $[(A \to C) \wedge (B \to C)] \to [(A \vee B) \to C]$

A6. $[A \wedge (B \vee C)] \to [(A \wedge B) \vee (A \wedge C)]$

Rules of inference:

Adjunction (Adj): $A\ \&\ B \Rightarrow A \wedge B$

Modus Ponens (MP): $A \to B\ \&\ A \Rightarrow B$

Prefixing (Pref): $B \to C \Rightarrow (A \to B) \to (A \to C)$

Suffixing (Suf): $A \to B \Rightarrow (B \to C) \to (A \to C)$

Definition A.2. The positive logic $\text{B}_{\text{K}+}$ is the result of adding the rule K (or rule Veq) to B_+ [cf. 11] and references therein): $A \Rightarrow B \to A$ (Veq abbreviates 'Verum e quodlibet' — 'A true proposition follows from any proposition').

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References

[1] Béziau, J.-Y., "A new four-valued approach to modal logic", *Logique et Analyse*, 54, 213 (2011): 109–121.

[2] Brady, R. T., "Completeness proofs for the systems RM3 and BN4", *Logique et Analyse*, 25 (1982): 9–32.

[3] Brady, R. T. (ed.), *Relevant Logics and Their Rivals*, vol. II, Ashgate, Aldershot, 2003.

[4] De, M., and H. Omori, "Classical negation and expansions of Belnap-Dunn logic", *Studia Logica*, 103, 4 (2015): 825–851. DOI: 10.1007/s11225-014-9595-7

[5] Kamide, N., and H. Omori, "An extended first-order Belnap-Dunn logic with classical negation", pages 79–93 in A. Baltag, J. Seligman and T. Yamada (eds.), *Logic, Rationality, and Interaction*, Springer, 2017. DOI: 10.1007/978-3-662-55665-8_6

[6] Łukasiewicz, J., *Aristotle's Syllogistic from the Standpoint of Modern Formal Logic*, Clarendon Press, Oxford, 1951.

[7] Łukasiewicz, J., "A system of modal logic", *The Journal of Computing Systems*, 1 (1953): 111–149.

[8] Méndez, J. M., and G. Robles, "A strong and rich 4-valued modal logic without Łukasiewicz-type paradoxes", *Logica Universalis*, 9, 4 (2015): 501–522. DOI: 10.1007/s11787-015-0130-z

[9] Méndez, J. M., G. Robles and F. Salto, "An interpretation of Łukasiewicz's 4-valued modal logic", *Journal of Philosophical Logic*, 45, 1 (2016), 73–87. DOI: 10.1007/s10992-015-9362-x

[10] Robles, G., and J. M. Méndez, "A binary Routley semantics for intuitionistic De Morgan minimal logic H_M and its extensions", *Logic Journal of the IGPL*, 23, 2 (2015): 174–193. DOI: 10.1093/jigpal/jzu029

[11] Robles, G., and J. M. Méndez, *Routley-Meyer Ternary Relational Semantics for Intuitionistic-Type Negations*, Elsevier, 2018.

[12] Robles, G., S. M. López, J. M. Blanco, M. M. Recio and J. R. Paradela, "A 2-set-up Routley-Meyer semantics for the 4-valued relevant logic E4", *Bulletin of the Section of Logic*, 45, 2 (2016): 93–109. DOI: 10.18778/0138-0680.45.2.03

[13] Routley, R., R. K. Meyer, V. Plumwood and R. T. Brady, *Relevant Logics and their Rivals*, vol. 1, Atascadero, CA: Ridgeview Publishing Co., 1982.

[14] Slaney, J. K., *MaGIC, Matrix Generator for Implication Connectives: Version 2.1, Notes and Guide*, Canberra: Australian National University, 1995. http://users.cecs.anu.edu.au/jks/magic.html (27/01/2021).

[15] Zaitsev, D., "Generalized relevant logic and models of reasoning" (doctoral dissertation), Moscow State Lomonosov University, 2012.

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